Powers of Monomial Ideals and the Ratliff-Rush Operation

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1) Setup

- We work in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$;
- each monomial ideal I of R has a unique minimal monomial generating set $G(I)$;
- all the considered objects will be $\mathfrak{m}$-primary monomial ideals of R, that is, monomial ideals I such that for some positive integers $d_1, \dots, d_n$ we have $\{x_1^{d_1}, \dots, x_n^{d_n}\} \subseteq G(I)$;
- we will not distinguish between monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and point $(\alpha_1, \dots, \alpha_n)$; multiplication of monomials corresponds to addition of points.

2) Boxes and good ideals

Definition 1. Let I be an ideal. Recall that $\{x_1^{d_1}, \dots, x_n^{d_n}\} \subseteq G(I)$ for some d_i. Let $a_1, \dots, a_n$ be nonnegative integers and denote

$$B_{a_1,\dots,a_n} := ([a_1 d_1, (a_1+1)d_1] \times \dots \times [a_n d_n, (a_n+1)d_n]) \cap \mathbb{N}^n.$$

$B_{a_1,\dots,a_n}$ will be called the **box** with coordinates $(a_1, \dots, a_n)$, associated to I.

Definition 2. We will say that an ideal I is **good** if the following holds: for every positive integer l, every minimal generator of I^l belongs to some box $B_{a_1,\dots,a_n}$ such that $a_1 + \dots + a_n = l - 1$.

3) Example of a bad ideal

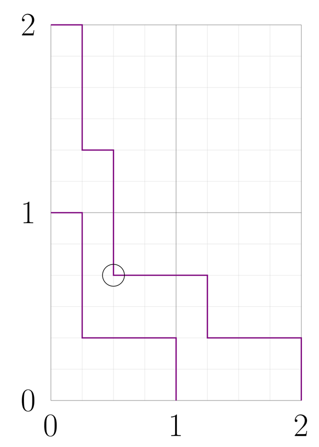

Fig. 1: powers of I: I and I^2

Example 3. Let $I = \langle x^4, y^6, xy^2 \rangle \subset \mathbb{K}[x, y]$. In this case the associated boxes have sizes 4×6. Then $I^2 = \langle x^8, x^5y^2, x^2y^4, xy^8, y^{12} \rangle$. We see that x^2y^4 only belongs to $B_{0,0}$. Therefore, the box decomposition principle fails already in I^2 and I is a bad ideal.

4) Example of a good ideal

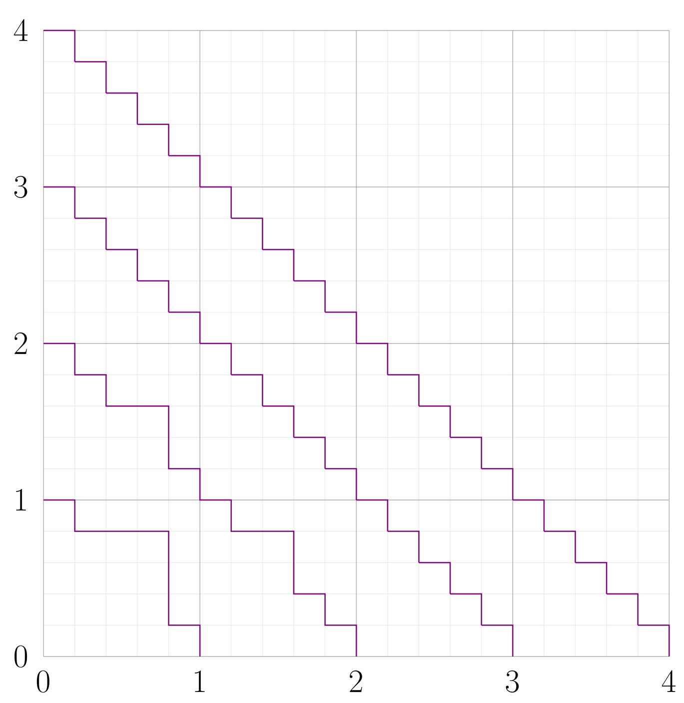

Fig. 2: powers of I: I, I^2, I^3 and I^4

Example 4. Let $I = \langle x^5, y^5, xy^4, x^4y \rangle \subset \mathbb{K}[x, y]$. In this case the associated boxes have sizes 5×5.

We see that:

all elements of $G(I)$ lie in $B_{0,0}$;

all elements of $G(I^2)$ lie in $B_{1,0}$ and $B_{0,1}$;

all elements of $G(I^3)$ lie in $B_{2,0}$, $B_{1,1}$ and $B_{0,2}$.

The pattern repeats in all powers of I: for every $l \geq 1$ each minimal generator of I^l belongs to some box whose sum of coordinates is $l - 1$. Therefore, I is a good ideal.

5) Ideals inside boxes and their connection to each other

Let $I = \langle x^5, y^5, xy^4, x^4y \rangle \subset \mathbb{K}[x, y]$ as in Example 4. Consider the box $B_{1,0}$. If we zoom into this box, we will see the following picture:

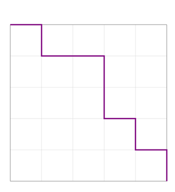

Fig. 3: $I_{1,0}$

Define $I_{1,0} := \langle y^5, xy^4, x^3y^2, x^4y, x^5 \rangle$. In other words, we pretend that the origin is the lower left corner of the box. We similarly define $I_{a_1,\dots,a_n}$ for any n, a good ideal I in R and nonnegative integers $a_1, \dots, a_n$.

Theorem 5. *Let I be a good ideal and let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be nonnegative integers such that $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$. Then $I_{a_1,\dots,a_n} \subseteq I_{b_1,\dots,b_n}$.*

6) Cones

Definition 6. Let $a_1, \dots, a_n$ be nonnegative integers. We will use the following notation:

$$C_{\underline{a_1},\underline{a_2},\dots,\underline{a_k},a_{k+1},a_{k+2},\dots,a_n} := \{(b_1, \dots, b_n) \in \mathbb{N}^n \mid b_1 = a_1, \dots, b_k = a_k, b_{k+1} \geq a_{k+1}, \dots, b_n \geq a_n\}.$$

We will use a similar notation for any configuration of fixed and non-fixed coordinates. Sets of this type will be called **cones**, for any cone the number of non-fixed coordinates will be called its **dimension** and $(a_1, \dots, a_n)$ will be called its **vertex**. Note that $\mathbb{N}^n = C_{0,0,\dots,0}$.

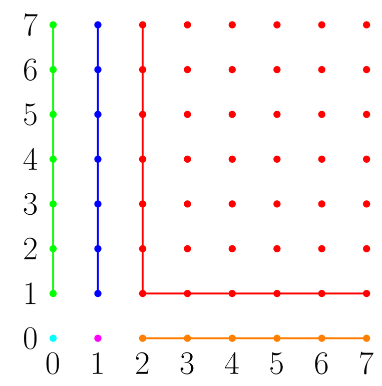

Fig. 4: a covering of $\mathbb{N}^2$ with cones

Example 7. Figure 4 represents the following six cones: $C_{\underline{0},\underline{0}}$, $C_{\underline{0},1}$, $C_{1,\underline{0}}$, $C_{\underline{1},\underline{1}}$, $C_{2,\underline{0}}$, $C_{2,1}$. The boundary lines are only drawn for better visibility. Clearly, the number of boundary lines equals the dimension of the cone.

7) Painting boxes of a good ideal

Theorem 8. *For any good ideal I there exists a finite coloring of $\mathbb{N}^n$ such that if $(a_1, \dots, a_n)$ has the same color as $(b_1, \dots, b_n)$, then $I_{a_1,\dots,a_n} = I_{b_1,\dots,b_n}$ and for each color the set of points of this color forms a cone.*

Sketch of the proof: First of all, note that it is possible to find a point $(a_1, \dots, a_n)$ such that the following holds: for all $(b_1, \dots, b_n) \geq (a_1, \dots, a_n)$ we have $I_{a_1,\dots,a_n} = I_{b_1,\dots,b_n}$. Indeed, if we assume the converse, then from Theorem 5 for every point of $\mathbb{N}^n$ there exists a strictly larger point that corresponds to a strictly larger ideal, therefore, we can build an infinite chain of strictly increasing ideals, which is impossible, for example, by Noetherianity of the polynomial ring. So existence of such a point $(a_1, \dots, a_n)$ is justified. Then we may paint all of $C_{a_1,\dots,a_n}$ with the same color. Now we have painted an n-dimensional "piece" of $\mathbb{N}^n$ and are left with finitely many "pieces" of dimensions $n - 1$ and less. We treat them similarly.

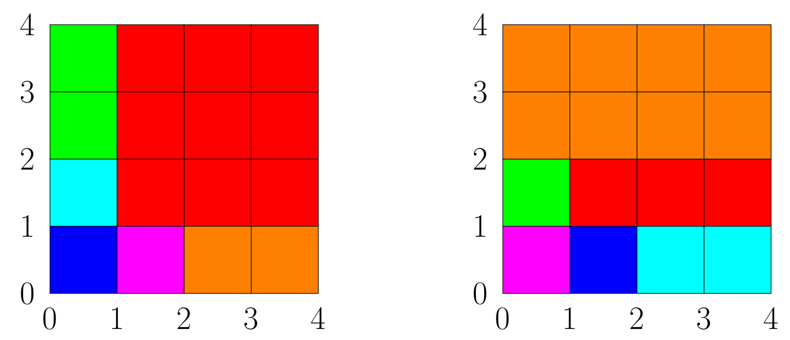

Fig. 5: two examples of possible colorings of $\mathbb{N}^2$, associated to I

Example 9. Let us look at Example 4 again. Figure 5 represents two possible colorings of boxes/points in $\mathbb{N}^n$, associated to I. Clearly, there are infinitely many colorings, but we will stick to one as soon as we found it: some further constructions depend on the coloring, but we do not want to put any additional indices.

8) The main result

Definition 10. Let R be any ring and I and J be ideals in it. The **quotient** of I and J is the ideal $I : J := \{r \in R \mid rJ \subseteq I\}$.

For a given ideal I we have an ascending chain of ideals $I^{k+1} : I^k$. If R is Noetherian, this chain must stabilize and the resulting ideal $\tilde{I} = \cup_{k\geq 0}(I^{k+1} : I^k)$ is called **the Ratliff-Rush closure** of I. Under some conditions on R and I, this ideal has some nice properties. In general it is not known how to compute $\tilde{I}$, but it is possible for good monomial ideals.

The idea is as follows. Given a good ideal I, we have a coloring as in Theorem 8. Note that we have fixed one coloring to work with. Our coloring is a disjoint union of cones. Each cone has a vertex. Let L denote the maximum of sums of coordinates of these vertices. For example, for both colorings in Figure 5 we have $L = 2$. The geometric meaning of this number is the following: starting from I^{L+1}, powers of I look similar to each other. For instance, let I be as in Example 4 and let us choose the left coloring in Figure 5. We know that every power of I starting from I^3 consists of a green box, an orange box and several red boxes and we exactly know where each of them is. This means, there is a pattern on high powers of I, and this is a key point for finding $\tilde{I}$.

Now let us consider the following line of boxes which is in bijection with nonnegative integer points on the x_1-axis: $B_{0,0,\dots,0}$, $B_{1,0,\dots,0}$, $B_{2,0,\dots,0}$ etc. Let $B_{q_1,0,\dots,0}$ be the stabilizing box of this sequence in a sense that q_1 is the smallest nonnegative integer such that $I_{t,0,\dots,0} = I_{q_1,0,\dots,0}$ for all $t \geq q_1$. Similarly, considering lines of boxes going along the other coordinate axes, we will get $q_2, q_3, \dots, q_n$.

Theorem 11. *Let I be a good ideal. Then $\tilde{I} = I_{q_1,0,\dots,0} \cap I_{0,q_2,\dots,0} \cap \dots \cap I_{0,\dots,0,q_n}$.*

Example 12. Let I be as in Example 4 together with the left coloring in Figure 5. Then $\tilde{I}$ equals the intersection of the ideal inside any of the orange boxes and the ideal inside any of the green boxes. They are both equal to $\langle x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5 \rangle$ and thus $\tilde{I} = \langle x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5 \rangle$.