An affine generalization of evacuation

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Evacuation

The Robinson–Schensted (RS) correspondence is a bijection between the symmetric group $\mathfrak{S}_n$ and pairs of standard Young tableaux of the same shape $\lambda \vdash n$.

Reflection of the type A_{n-1} Dynkin diagram induces an involution r on $\mathfrak{S}_n$; in one-line notation, r is the "reverse complement" map

$$w_1 \cdots w_n \mapsto (n+1-w_n) \cdots (n+1-w_1).$$

M.-P. Schützenberger showed that there is a shape-preserving involution e, called evacuation, such that if $w \overset{\mathrm{RS}}{\mapsto} (P, Q)$, then $r(w) \overset{\mathrm{RS}}{\mapsto} (e(P), e(Q))$.

Example: computing evacuation

$$\underset{P}{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6\\\cline{1-2} 5\\\cline{1-1}\end{array}} \xrightarrow{\text{rotate}} \begin{array}{ccc} &&5\\ &6&3\\ 4&2&1 \end{array} \xrightarrow{i\mapsto n+1-i} \begin{array}{ccc} &&2\\ &1&4\\ 3&5&6 \end{array} \xrightarrow{\text{slide}} \begin{array}{ccc} &1&2\\ &4&6\\ 3&5& \end{array} \xrightarrow{\text{slide}} \begin{array}{ccc} &1&2\\ 3&4&6\\ 5&& \end{array} \xrightarrow{\text{slide}} \underset{e(P)}{\begin{array}{ccc} 1&2&6\\ 3&4&\\ 5&& \end{array}}$$

Affine evacuation

Let $\widetilde{\mathfrak{S}}_n$ be the extended affine symmetric group, that is, the group of bijections $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i+n) = f(i) + n$ for all i. Such a bijection is called an (extended) affine permutation; it is determined by the "window" $[f(1), \ldots, f(n)]$.

Let λ be a partition of n. A tabloid of shape λ is a bijective filling of the Young diagram of λ with $\{1, \ldots, n\}$ such that rows are increasing. We write $\mathcal{T}(\lambda)$ for the set of tabloids of shape λ.

In 2015, M. Chmutov, P. Pylyavskyy, and E. Yudovina introduced the affine matrix-ball construction (AMBC), an injective map

$$\mathrm{AMBC} \ : \ \widetilde{\mathfrak{S}}_n \to \bigsqcup_{\lambda \vdash n} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \times \mathbb{Z}^{\ell(\lambda)}$$

where $\ell(\lambda)$ is the number of parts of λ. For example,

$$[2, -3, -2, 13, 0, 5] \overset{\mathrm{AMBC}}{\mapsto} \left(\begin{array}{ccc} 3&4&6\\ 2&5&\\ 1&& \end{array}, \begin{array}{ccc} 1&4&6\\ 3&5&\\ 2&& \end{array}, \begin{pmatrix} -2\\ 0\\ 1 \end{pmatrix} \right).$$

The image of AMBC consists of triples (P, Q, ρ) where the entries of the integer vector ρ satisfy linear inequalities which depend on the tabloids P and Q.

The reflection of the affine type A Dynkin diagram that fixes the affine node induces an involution r on $\widetilde{\mathfrak{S}}_n$; in window notation, r is the map

$$[w_1, \ldots, w_n] \mapsto [n+1-w_n, \ldots, n+1-w_1].$$

Theorem 1 (CFKLY)

There is a shape-preserving involution e on tabloids such that if $w \mapsto (P, Q, \rho)$, then $r(w) \mapsto (e(P), e(Q), \rho')$ for some ρ'. This map, which we call affine evacuation, agrees with usual evacuation when applied to a tabloid with increasing columns. In general, affine evacuation can be computed by an algorithm based on the combinatorial R-matrix.

The combinatorial R-matrix is a map $R : B^r \times B^s \to B^s \times B^r$, where B^k is the set of semistandard tableaux of shape $\langle k \rangle$. It is characterized by the property that $(a, b) \overset{R}{\mapsto} (a', b')$ if and only if the Schensted row insertion $(a \leftarrow b)$ equals the Schensted row insertion $(a' \leftarrow b')$. For example,

$$(\,2\,3\,5\,5\,,\ 1\,4\,) \overset{R}{\mapsto} (\,2\,5\,,\ 1\,3\,4\,5\,)$$

because

$$2\,3\,5\,5 \leftarrow 1\,4 = \begin{array}{cccc} 1&3&4&5\\ 2&5&& \end{array} = 2\,5 \leftarrow 1\,3\,4\,5.$$

Example: computing affine evacuation

$$\underset{P}{\begin{array}{ccc} 2&3&6\\ 1&5&\\ 4&& \end{array}} \xrightarrow{\text{flip}} \begin{array}{ccc} 4&&\\ 1&5&\\ 2&3&6 \end{array} \xrightarrow{i\mapsto n+1-i} \begin{array}{ccc} 3&&\\ 2&6&\\ 1&4&5 \end{array} \xrightarrow{R} \begin{array}{ccc} 3&&\\ 2&5&6\\ 1&4& \end{array} \xrightarrow{R} \begin{array}{ccc} 2&3&6\\ 5&&\\ 1&4& \end{array} \xrightarrow{R} \underset{e(P)}{\begin{array}{ccc} 2&3&6\\ 4&5&\\ 1&& \end{array}}$$

Fixed points of evacuation

Suppose $\lambda \vdash n$. Let χ^λ be the irreducible character of $\mathfrak{S}_n$ indexed by λ, and let

$$f^\lambda(q) = [n]! \prod_{c \in \lambda} \frac{1}{[h_c]} \quad \text{where} \quad [k] = \frac{1-q^k}{1-q} \quad \text{and} \quad [k]! = [k][k-1]\cdots[1]$$

be the q-analogue of the hook-length formula. Let w_0 be the long element of $\mathfrak{S}_n$.

Theorem A (J. Stembridge '96, R. Stanley '09)

Let $S(\lambda)$ be the set of standard tableaux of shape λ that are fixed by evacuation.

1. $\#S(\lambda) = f^\lambda(-1)$.
2. $\#S(\lambda) = (-1)^{(\lambda_2+\lambda_4+\ldots)} \chi^\lambda(w_0)$.
3. $S(\lambda)$ is in bijection with the set of standard domino tableaux of shape λ or $\lambda/\langle 1 \rangle$, depending on whether n is even or odd.

Fixed points of affine evacuation

For $w \in \mathfrak{S}_n$, the Green's polynomial $\mathcal{Q}^\lambda_w(q)$ is given by

$$\mathcal{Q}^\lambda_w(q) = \sum_{\mu \vdash n} \chi^\mu(w) \widetilde{K}_{\mu\lambda}(q),$$

where $\widetilde{K}_{\mu\lambda}(q) = \sum_{T \in \mathrm{SSYT}(\mu,\lambda)} q^{\mathrm{cocharge}(T)}$ is the cocharge Kostka–Foulkes polynomial. The class function $\mathcal{Q}^\lambda : \mathfrak{S}_n \to \mathbb{Z}[q]$ is (essentially) the graded character of the $\mathfrak{S}_n$-representation on the cohomology of the type A Springer fiber associated to λ.

Theorem 2 (CFKLY)

The number $t(\lambda)$ of self-evacuating tabloids of shape λ is given by

$$t(\lambda) = \mathcal{Q}^\lambda_{w_0}(-1).$$

For example, there are four self-evacuating tabloids of shape $\langle 2, 2 \rangle$:

$$\begin{array}{cc} 1&2\\ 3&4 \end{array} \qquad \begin{array}{cc} 1&3\\ 2&4 \end{array} \qquad \begin{array}{cc} 2&4\\ 1&3 \end{array} \qquad \begin{array}{cc} 3&4\\ 1&2 \end{array}.$$

On the other hand,

$$\begin{aligned} \mathcal{Q}^{\langle 2,2 \rangle}_{w_0}(q) &= \chi^{\langle 4 \rangle}(w_0) \widetilde{K}_{\langle 4 \rangle \langle 2,2 \rangle}(q) &&+ \chi^{\langle 3,1 \rangle}(w_0) \widetilde{K}_{\langle 3,1 \rangle \langle 2,2 \rangle}(q) &&+ \chi^{\langle 2,2 \rangle}(w_0) \widetilde{K}_{\langle 2,2 \rangle \langle 2,2 \rangle}(q) \\ &= 1 \cdot 1 &&+ (-1) \cdot q &&+ 2 \cdot q^2, \end{aligned}$$

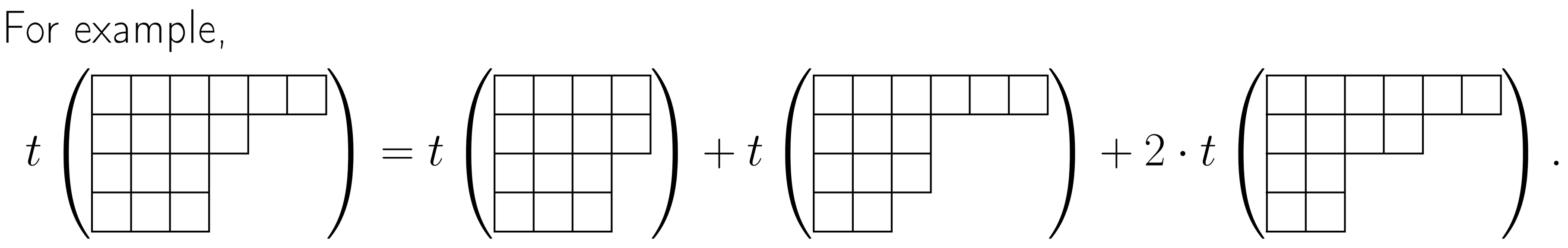
so $\mathcal{Q}^{\langle 2,2 \rangle}_{w_0}(-1) = 1 + 1 + 2 = 4$.

Theorem 3 (CFKLY)

Suppose that $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots \rangle$. Let $\lambda\downarrow^{(i)}_{(i-2)}$ be the partition formed by replacing a part of size i with one of size $i-2$, and let $\lambda\downarrow^{(i,i)}_{(i-1,i-1)}$ be the partition formed by replacing two parts of size i with two parts of size $i-1$. Then $t(\lambda)$, the number of self-evacuating tabloids of shape λ, satisfies the "domino-like" recurrence relation

$$t(\lambda) = \sum_{\substack{i \,:\, i \geq 2, \\ m_i \text{ is odd}}} t\left(\lambda\downarrow^{(i)}_{(i-2)}\right) + \sum_i 2 \left\lfloor \frac{m_i}{2} \right\rfloor \cdot t\left(\lambda\downarrow^{(i,i)}_{(i-1,i-1)}\right).$$

For example,



Remarks

- To prove Theorem 2, we use RSK and Theorem A(2) to reduce to the evaluation of the Kostka–Foulkes polynomials at $q = -1$, and we prove this evaluation using the theory of rigged configurations.
- D. Kim has shown that for certain λ, $\mathcal{Q}^\lambda_{w_0}(-1)$ is equal to the Euler characteristic of a type B or C Springer fiber.
- The proof of the recurrence in Theorem 3 uses Theorem 2 and an argument involving symmetric functions due to D. Kim. It is an open problem to find a bijective proof of this recurrence; such a proof would perhaps give rise to a definition of "domino tabloids."

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