Skew Key Polynomials and the Key Poset

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Abstract

We generalize Young's lattice on integer partitions to a new partial order on weak compositions called the ***key poset***. Saturated chains in this poset correspond to ***standard key tableaux***, the combinatorial objects that generate the key polynomial basis for the polynomial ring, a generalization of the Schur basis for symmetric functions. Generalizing skew Schur functions, we define ***skew key polynomials*** in terms of the poset, and, using weak dual equivalence, we give a nonnegative ***weak composition Littlewood–Richardson rule*** for the key expansion of skew key polynomials.

Main Result

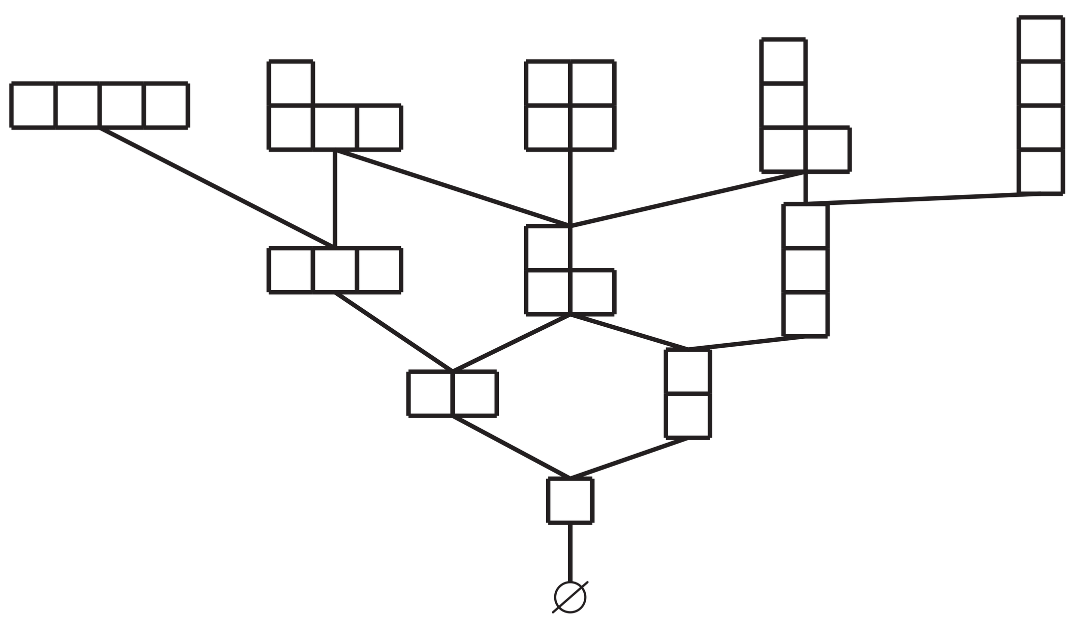
We define ***skew key polynomials*** for ***weak compositions*** $\mathbf{a} \prec \mathbf{d}$ in the key poset. Then

$$\kappa_{\mathbf{d}/\mathbf{a}} = \sum_{\mathbf{b}} c^{\mathbf{d}}_{\mathbf{a},\mathbf{b}} \kappa_{\mathbf{b}}$$

where $c^{\mathbf{d}}_{\mathbf{a},\mathbf{b}}$ are ***nonnegative*** integers.

Young's Lattice

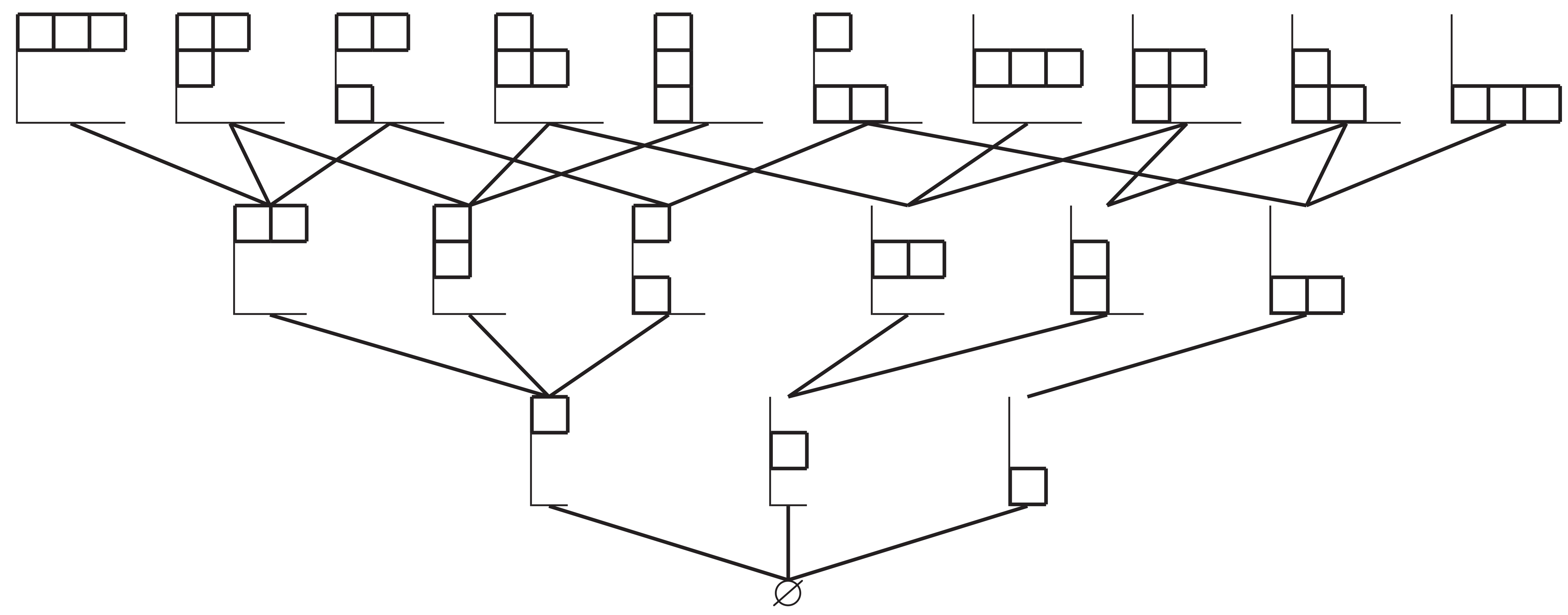
Young's lattice is the partial order $\subseteq$ on ***partitions*** given by containment of diagrams, that is $\lambda \subseteq \mu$ if and only if $\lambda_i \leq \mu_i$ for all i.



The ***cover relations for Young's lattice*** are $\lambda \prec\!\!\cdot\, \mu$ if and only if μ is obtained from λ by adding a single box to the end of a row for which the higher row is strictly shorter.

Key Poset

The ***key poset*** is the partial order $\prec$ on ***weak compositions*** of length n defined by the relation $a \preceq b$ if and only if $a_i \leq b_i$ for $i = 1, 2, \ldots, n$ and for any indices $1 \leq i < j \leq n$ for which $b_j > a_j$ and $a_i > a_j$, we have $b_i > b_j$. This is *not* equivalent to containment of diagrams.



The ***cover relations for the key poset*** are $a \prec\!\!\cdot\, b$ if and only if b is obtained from a by incrementing a_j by 1 where for any $i < j$ we have $a_i \neq a_j + 1$.

Schur Functions

A ***standard Young tableau of shape*** λ is a bijective filling of λ with $1, 2, \ldots, n$ such that row entries increase left to right and column entries increase bottom to top.

A standard Young tableau is equivalent to a ***saturated chain*** in Young's lattice.

The ***skew Schur function indexed by*** $\lambda \subset \nu$ is

$$s_{\mu/\lambda}(X) = \sum_{T \in \mathrm{SYT}(\nu/\lambda)} F_{\mathrm{Des}(T)}(X)$$

where $F_\alpha(X)$ is a ***fundamental quasisymmetric function*** [5]. For example,

$$s_{(2,1)}(X) = F_{(2,1)}(X) + F_{(1,2)}(X).$$

Key Polynomials

Definition 1 ([3]). A ***standard key tableau*** is a bijective filling of a key diagram with $1, 2, \ldots, n$ such that rows weakly decrease from left to right, and if some entry i is above and in the same column as an entry k with $i < k$, then there is an entry j immediately right of k and $i < j$.

Theorem 1 (A.-vW.[2]). *Saturated chains from $\varnothing$ to a in the key poset are in bijection with standard key tableaux of shape a by the correspondence placing $n - i + 1$ into the unique cell of $a^{(i)}/a^{(i-1)}$.*

The ***key polynomial indexed by*** $\mathbf{a}$ is

$$\kappa_a = \sum_{T \in \mathrm{SKT}(a)} \mathfrak{F}_{\mathrm{des}(T)}$$

where $\mathfrak{F}_{\mathbf{a}}$ is a ***fundamental slide polynomial*** [1]. For example,

$$\kappa_{(0,2,1)} = \mathfrak{F}_{(0,2,1)} + \mathfrak{F}_{(1,2,0)}.$$

Using the key poset we define ***skew key polynomials*** for weak compositions $\mathbf{a} \prec \mathbf{d}$ by

$$\kappa_{d/a} = \sum_{T \in \mathrm{SKT}(d/a)} \mathfrak{F}_{\mathrm{des}(T)}.$$

This generalizes the ***flagged skew Schur polynomials*** studied by Reiner–Shimozono [6].

Schur Positivity

Definition 2. The ***Littlewood–Richardson coefficients*** $c^{\nu}_{\lambda,\mu}$ are given by

$$s_{\nu/\lambda}(X) = \sum_{\mu} c^{\nu}_{\lambda,\mu} s_\mu(X)$$

Theorem 2. *For all λ, μ, ν, we have $c^{\nu}_{\lambda,\mu} \in \mathbb{N}$.*

One of myriad proofs uses ***dual equivalence*** [4] to consolidate skew standard Young tableaux into equivalence classes, each corresponding to a single Schur function.

Key Positivity

Definition 3. The ***weak composition Littlewood–Richardson coefficients*** $c^{\mathbf{d}}_{\mathbf{a},\mathbf{b}}$ are given by

$$\kappa_{\mathbf{d}/\mathbf{a}} = \sum_{\mathbf{b}} c^{\mathbf{d}}_{\mathbf{a},\mathbf{b}} \kappa_{\mathbf{b}}$$

Theorem 3 (A.-vW. [2]). *For $\mathbf{a} \prec \mathbf{d}$ in the key poset, we have $c^{\mathbf{d}}_{\mathbf{a},\mathbf{b}} \in \mathbb{N}$ for all $\mathbf{b}$.*

Our proof utilizes ***weak dual equivalence*** [3] to consolidate skew standard key tableaux into equivalence classes, each corresponding to a single key polynomial.

Moreover, this result is tight. We could define skew key polynomials for any $\mathbf{a} \subset \mathbf{d}$.

Theorem 4 (A.-vW. [2]). *For $\mathbf{a} \subset \mathbf{d}$ s.t. $\mathbf{a} \not\prec \mathbf{d}$ in the key poset, $c^{\mathbf{d}}_{\mathbf{a},\mathbf{b}} < 0$ for some weak composition $\mathbf{b}$.*

References

[1] S. Assaf and D. Searles. Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams. *Adv. Math.*, 306:89–122, 2017.

[2] S. Assaf and S. van Willigenburg. Skew key polynomials and a generalized Littlewood–Richardson rule. arXiv:1905.11526.

[3] S. H. Assaf. Weak dual equivalence for polynomials. arXiv:1702.04051.

[4] S. H. Assaf. Dual equivalence graphs I: A new paradigm for Schur positivity. *Forum Math. Sigma*, 3:e12, 33, 2015.

[5] I. M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.

[6] V. Reiner and M. Shimozono. Key polynomials and a flagged Littlewood-Richardson rule. *J. Combin. Theory Ser. A*, 70(1):107–143, 1995.

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