Equidistribution of Weierstrass points on a tropical curve

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Weierstrass points

The **Weierstrass locus** $W(D)$ of a divisor D on an algebraic curve X consists of points of "higher than expected tangency" with hyperplanes in the projective embedding $\phi_D : X \to \mathbb{P}^r$, i.e.

$$W(D) = \{x \in X : \phi(X) \cap H \geq (r+1)x \text{ for some hyperplane } H\}.$$

On a genus 1 curve, these are the N-torsion points (up to some choice of 0) where the divisor has degree N.

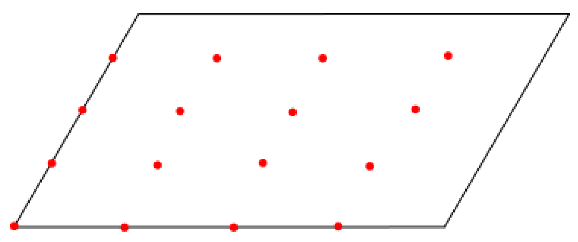

Figure 1. A complex elliptic curve with its 4-torsion points

As $N \to \infty$, N-torsion points "evenly distribute" over a complex elliptic curve.

In general, Mumford suggested we should consider Weierstrass points as *higher-genus analogues of N-torsion points*. This makes it natural to ask:

Problem

How do Weierstrass points distribute on a curve?

For curves over $\mathbb{C}$, this problem was answered by Amnon Neeman.

Theorem (Neeman, 1984)

If X is complex algebraic curve, the Weierstrass points $W(D_N)$ distribute according to the Bergman measure on X as $N \to \infty$.

We can also consider algebraic curves over a non-Archimedean field ($\mathbb{K}$, val: $\mathbb{K}^\times \to \mathbb{R}$) which we assume is algebraically closed, e.g. $\mathbb{K} = \cup_{n\geq 1}\mathbb{C}((t^{1/n}))$. The Weierstrass points $W(D)$ lie in $X(\mathbb{K}) \subset X^{\text{an}}$.

Theorem (Amini, 2014)

If X^{an} is Berkovich curve, the Weierstrass points $W(D_N)$ distribute according to the Zhang measure on X^{an} as $N \to \infty$.

Zhang's canonical measure

We consider Γ a **resistor network** making each edge a resistor with resistance = length. Given points $y, z \in \Gamma$, we define $j_z^y : \Gamma \to \mathbb{R}$ by

$$j_z^y = \left(\begin{array}{l} \text{voltage on } \Gamma \text{ when 1 unit of} \\ \text{current is sent from } y \text{ to } z \end{array} \right)$$

and Γ is "grounded" at z. The **current** through an edge is the slope $|j'|$ of the voltage function (Ohm's law). The **canonical measure** $\mu(e)$ is the "current defect"

$$\mu(e) = 1 - (\text{ current through } e \text{ when 1 unit sent from } e^- \text{ to } e^+ \text{ }).$$

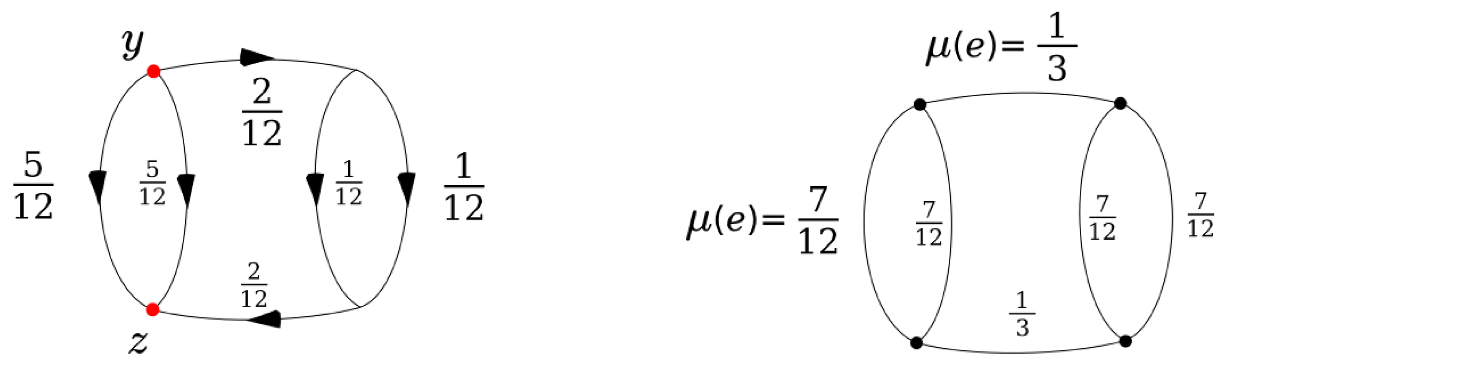

Figure 4. Current flow from y to z on Γ with unit edge lengths, and canonical measures on Γ

Theorem (R. 2019)

(i) On a tropical curve Γ, $W(D)$ is finite for a *generic* divisor class $[D]$.

(ii) Let $[D_N]$ be a generic divisor class of degree N. Then the Weierstrass points $W(D_N)$ distribute according to the Zhang measure μ on Γ as $N \to \infty$.

Proof idea:

$$\begin{array}{ccc} \text{(discrete current flow)} & \xrightarrow{N\to\infty} & \text{(continuous current flow)} \\ \updownarrow & & \updownarrow \\ \#(W(D_N) \cap e) & & \text{canonical measure } \mu(e) \end{array}$$

Tropical curves

A **tropical curve** is a metric space Γ obtained from a finite graph by assigning positive, real edge lengths. The **genus** of Γ is $g = \dim H_1(\Gamma, \mathbb{R})$.

An (effective) **divisor** D on Γ is a finite collection of "chips" placed on Γ. **Linear equivalence** means we may move any subset of chips along a cut-set of Γ, at the same speed and direction. Intuitively, this amounts to "discrete current flow".

A **reduced divisor** $\text{red}_q[D]$ is the unique representative linearly equivalent to D whose chips are "as close as possible" to $q \in \Gamma$.

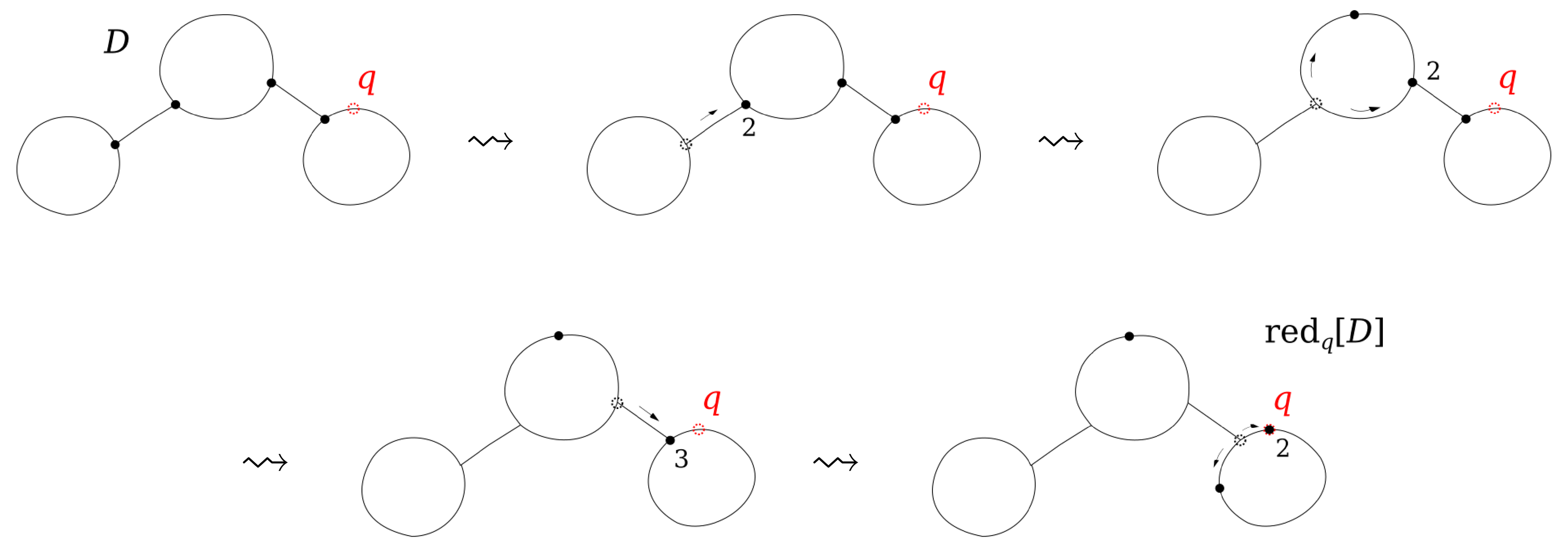

Figure 2. Reduced divisor $\text{red}_q[D]$ linearly equivalent to D

algebraic curve X		tropical curve Γ
divisors $\text{Div}(X)$	$\rightsquigarrow$	divisors $\text{Div}(\Gamma)$
meromorphic functions	$\rightsquigarrow$	piecewise $\mathbb{Z}$-linear functions
linear system $\lvert D\rvert = \mathbb{P}^r$	$\rightsquigarrow$	$\lvert D\rvert$ = polyhedral complex of dim $\geq r$
rank $r = \dim \lvert D\rvert$	$\rightsquigarrow$	rank r = Baker-Norine rank

Table 1. Divisor theory from algebraic curves to tropical curves

Tropical Weierstrass points

The **tropical Weierstrass locus** $W(D)$ of a divisor on a metric graph Γ is

$$W(D) = \{x \in \Gamma : \ E \geq (r+1)x \quad \text{for some } E \in |D|\}$$

where $r = \deg(D) - g$ when $\deg(D) \geq 2g-1$ (r is the Baker-Norine rank). Equivalently, using reduced divisors

$$x \in W(D) \quad \Leftrightarrow \quad \text{red}_x[D] \geq (r+1)x.$$

In contrast to algebraic curves, $W(D)$ is *not always finite* on Γ.

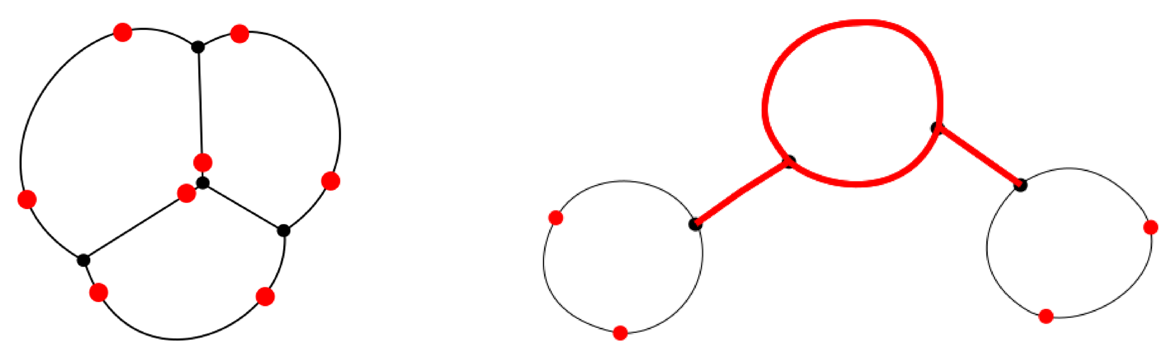

Figure 3. Weierstrass locus $W(K)$ on two genus 3 curves

Break divisors and stability

The break divisor construction due to Mikhalkin and Zharkov can be used to fix non-finiteness of $W(D)$. A **break divisor** is a divisor such that $\Gamma \setminus D$ is a tree*. This gives a canonical effective representative for each divisor class $[D]$ of degree g.

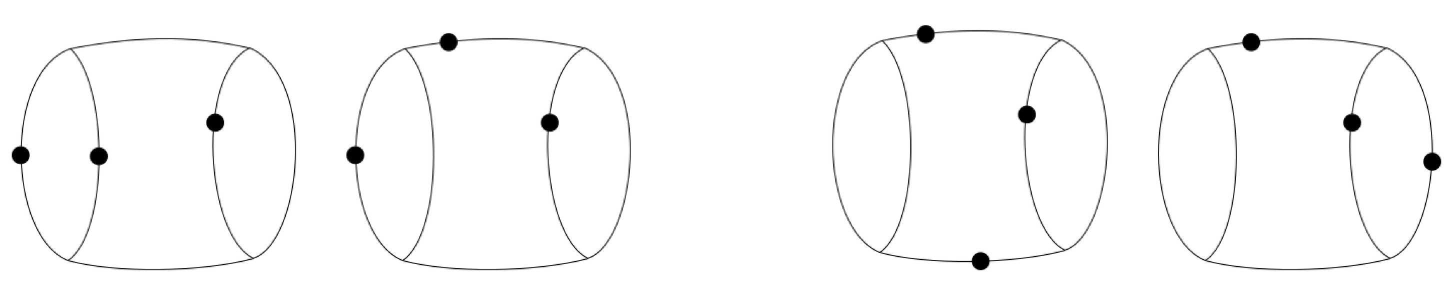

Figure 5. Break divisors (left) and non-break divisors (right)

We define the **stable tropical Weierstrass locus** $W_{\text{st}}(D)$ as

$$W_{\text{st}}(D) = \{x \in \Gamma : \ \text{br}[D - (N-g)x] \geq x\} \qquad \text{where } \deg(D) = N \geq g.$$

Theorem (R. 2019)

(i) If D has degree $N \geq 2g-1$ (or D is "nonspecial"), then $W_{\text{st}}(D) \subset W(D)$.

(ii) For any divisor D of degree $N \geq g$, $W_{\text{st}}(D)$ is finite and has cardinality

$$\#W_{\text{st}}(D) \leq g(N-g+1).$$

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