Explicit Formulas for Characters of the Module of Multivariate Diagonal Harmonics

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Introduction

- The Character of the Module of Multivariate Diagonal Harmonic Polynomials here takes the form:

$$\mathcal{E}_{rn,n} = \sum_{\mu}\sum_{\lambda} c_{\lambda,\mu} \underbrace{s_\lambda}_{GL_k} \otimes \underbrace{s_\mu}_{\mathbb{S}_n}, \text{ or equivalently, } \langle \mathcal{E}_{rn,n}, s_\mu\rangle = \sum_\lambda c_{\lambda,\mu} s_\lambda$$

For example, when $n = 4$ we have:

$$\mathcal{E}_{4,4} = 1\otimes s_4 + (s_1+s_2+s_3)\otimes s_{31} + (s_2+s_{21}+s_4)\otimes s_{22} + (s_{11}+s_{21}+s_{31}+s_3+s_4+s_5)\otimes s_{211} + (s_{111}+s_{31}+s_{41}+s_6)\otimes s_{1111}$$

- We give explicit combinatorial formulas for $\langle \mathcal{E}_{rn,n}, s_\mu\rangle|_{\text{hooks}}$, which explain all the terms in the above special case since $\mathcal{E}_{4,4} = \mathcal{E}_{4,4}|_{\text{hooks}}$.
- Note that the formula is proven only for certain shape μ, therefore in the previous example the proof dose not account for the part $(s_2+s_{21}+s_4)\otimes s_{22}$.
- These are characters of "rational" $GL_k \times \mathbb{S}_n$-modules. The modules are introduced in [1], as a generalization of diagonal harmonics ($k = 2$ & $r = 1$).

New Combinatorial Object $T_{n,s}$

Consider a path of *North and East steps* of length $n-2-s$ that starts at height s.

The set of such paths is $T_{n,s}$. The **area** of a path is the number of boxes to the south-east of the path. The **height** of a path is the height of its end point.

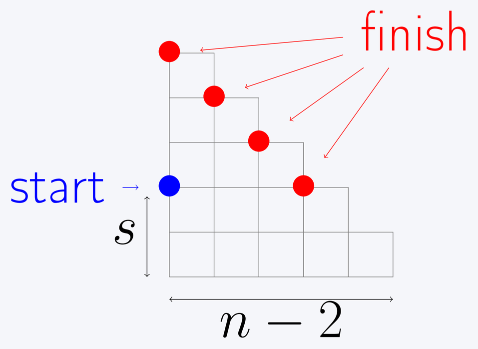

Figure 1 : $T_{7,2}$

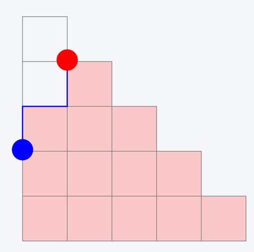

Figure 2 : $\text{area}(NEN) = 13$

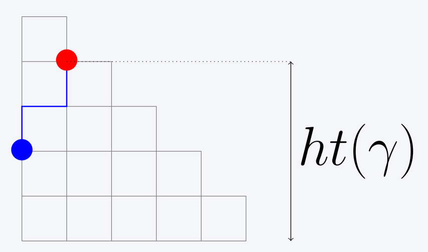

Figure 3 : $\text{ht}(NEN) = 4$

Proposition *(see extended abstract, proofs in [3])*:

Let $T_{n,s}(q,z) = \sum_{\gamma\in T_{n,s}} q^{\text{area}(\gamma)} z^{\text{ht}(\gamma)}$. Then for $r = n-s-2$ we have:

$$T_{n,s}(q,z) = \sum_{j=0}^{r} q^{\binom{s+j+1}{2}+s(r-j)} \begin{bmatrix} r \\ j\end{bmatrix}_q z^{j+s} = T_{r+2,0}(q,z)z^s q^{rs+\binom{s+1}{2}}$$

In particular, if $s = 0$ then $T_{n,0}(q,z) = \sum_{j=0}^{n-2} q^{\binom{j+1}{2}} \begin{bmatrix} n-2 \\ j\end{bmatrix}_q z^j = (-qz;q)_{n-2}$

From alternating sum to positive sum

Proposition *(see extended abstract, proofs in [3])*: The following equality holds:

$$\sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1} (-1)^k \begin{bmatrix} n-1 \\ j+k\end{bmatrix}_q q^{-k+\binom{j+k+1}{2}} z^{j-1} = T_{n,0}(q,qz)q.$$

Remark: For more general versions and results, see [3].

Sketch of proof:

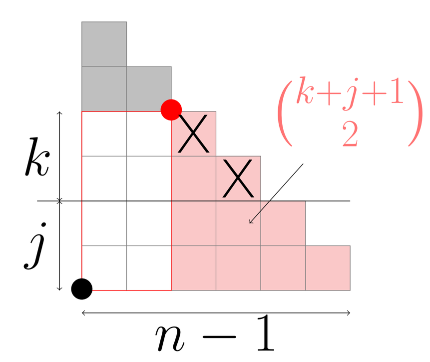

Figure 4 : Representation of the term $\begin{bmatrix} n-1 \\ j+k\end{bmatrix}_q q^{-k+\binom{k+j+1}{2}}$ in the last equation

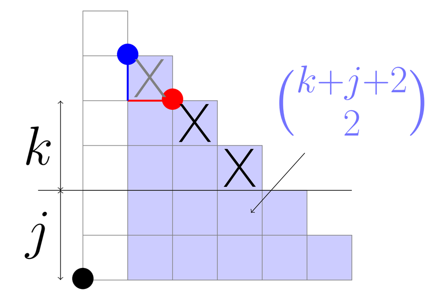

Figure 5 : Comparing canceling terms. i.e. path ending with a North step in $\mathcal{C}^{n-1}_{j+k+1}$ and with an east step in $\mathcal{C}^{n-1}_{j+k}$

Future work

- Find a formula for $\nabla(e_n)$ or $\nabla(C_\alpha)$ in terms of $s_\lambda \otimes s_\mu$
- Multivariate lift of previous formula.
- Writing the Schröder paths in terms of Schur functions in q and t. Equivalently crystal decomposition of the Schröder paths. (Partial results in preparation.)
- Bijections between $T_{n,s}$ and other combinatorial objects. Some objects are refined by $T_{n,s}$ (in preparation).

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Formula for $\langle \mathcal{E}_{rn,n}, s_\mu\rangle|_{\text{hooks}}$

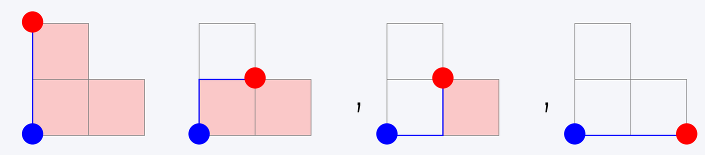
Main theorem *(see extended abstract, proof in [3])*:

If $r = 1$ and $\mu \in \{(n), (n-1,1), (n-2,1,1), (1^n)\}$ then:

$$\langle \mathcal{E}_{rn,n}, s_\mu\rangle|_{\text{hooks}} = \sum_{\tau\in\text{SYT}(\mu)}\sum_{\gamma\in T_{n,\text{des}(\tau')}} s_{\text{hook}(\gamma)}(q_1,q_2,q_3,\ldots),$$

where $\text{hook}(\gamma) = \left((r-1)\binom{n}{2} + \text{area}(\gamma) + \text{ht}(\gamma) - \text{maj}(\tau') + 1, 1^{n-2-\text{ht}(\gamma)}\right)$.

The elements of $T_{2,0}$:



area(γ) :	3	2	1	0
ht(γ) :	2	1	1	0
hook(γ) :	$6,1^0$	$4,1^1$	$3,1^1$	$1,1^2$

Figure 6 : Example of $\langle \mathcal{E}_{4,4}, e_4\rangle|_{hooks} = s_6 + s_{41} + s_{31} + s_{111}$
Example of $\langle \mathcal{E}_{16,4}, e_4\rangle|_{hooks} = s_{24} + s_{22,1} + s_{21,1} + s_{19,1,1}$

- In [1] F. Bergeron showed that for $r = 1$, $\langle \mathcal{E}_{rn,n}, s_{k+1,1^{n-k-1}}\rangle = e_k^\perp\langle \mathcal{E}_{rn,n}, s_{1^n}\rangle$ for all k. When $\mu = 1^n$, the theorem holds for all r satisfying this identity for all k.
- The theorem is a lift of the formula $\Delta'_{e_{n-k-1}} e_n|_{t=0}$, by Haglund, Rhoades and Shimozono [2]. Therefore, the theorem holds for all μ when $r = 1$, if F. Bergeron's conjecture $e_k^\perp\langle \mathcal{E}_{n,n}, s_\mu\rangle =_{\langle 2\rangle} \langle \Delta'_{e_{n-k-1}} e_n, s_\mu\rangle$ is true for all k.
- In [3], we offer a more general method of lifting formulas to the multivariate case.
- Moreover, in [3], we show how to do $\langle \mathcal{E}_{n,n}, s_{k+1,1^{n-k-1}}\rangle = e_k^\perp\langle \mathcal{E}_{n,n}, s_{1^n}\rangle$ directly in terms of paths. This gives a lower bound on the coefficients.

A lift of the Delta Conjecture

By construction the restriction to 2 variables contains the $\Delta'_{e_{n-k-1}} e_n|_{t=0}$ of Haglund, Rhoades and Shimozono presented in [2].

Restriction to 1 variable is viewed as a restriction to shapes that have one part.

Proposition *(see extended abstract, proof in [3])*: For all k we have:

$$\langle \Delta'_{e_k}(e_n), e_n\rangle|_{\text{hooks}} = \sum_{\tau\in\text{SYT}((n-k,1^k))} s_{\text{maj}(\tau)}(q,t) + \sum_{i=2}^{k} s_{\text{maj}(\tau)-i,1}(q,t).$$

If $r = 1$ and $\mu \in \{(k,1^{n-k}) \mid 1\le k\le n\}$ or if $r > 1$ and $\mu = 1^n$, then:

$$\langle \nabla^r(e_n), s_\mu\rangle|_{\text{hooks}} = \sum_{\tau\in\text{SYT}(\mu)} s_{r\binom{n}{2}-\text{maj}(\tau')}(q,t) + \sum_{i=2}^{\text{des}(\tau)} s_{r\binom{n}{2}-\text{maj}(\tau')-i,1}(q,t).$$

- **Example:** $\langle \nabla^4(e_4), e_4\rangle|_{hooks} = s_{24}(q,t) + s_{22,1}(q,t) + s_{21,1}(q,t)$
- When $q = 0$ or $t = 0$ the equation above holds for arbitrary μ if $r = 1$, and for $r > 1$ if μ is hook shaped.
- Formulas for $e^\perp_{n-k-1}(\langle \mathcal{E}_{n,n}, s_\mu\rangle|_{\text{hooks}})^{\langle 2\rangle}$ and $\langle \Delta'_{e_k}(e_n), s_\mu\rangle|_{\text{some hooks}}$ available in [3]

References

[1] F. Bergeron.
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[2] J. Haglund, B. Rhoades, and M. Shimozono.
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[3] N. Wallace.
Explicit description of some irreducible components of the $(\text{GL}_\infty \times \mathbb{S}_n)$-modules of multivariate diagonal harmonic polynomials, 2019.

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