Definitions with examples: Split Networks and Level-1 Phylogenetic Networks.

(1) A split network \( s \), which represents the collection of non-trivial splits \( A|B \) in the split system (2). The same system is represented by (3) the subdivided polygon. Next there is pictured (4) a level-1 phylogenetic network \( N \) as defined in [3], which is found by forgetting the internal structure of \( s \) via the map \( L \). From any level-1 network we can recover a split network \( \Sigma(N) \) which has all the splits corresponding to minimal cuts in \( N \) as shown in two representations: (5) and (6).

Definition: New Polytopes.

The vector \( x(s) \) is defined to have lexicographically ordered components \( x_{ij}(s) \) for each unordered pair of distinct leaves \( i, j \in [n] \) as follows:

\[
x_{ij}(s) = \begin{cases} 2^{k-b_{ij}} & \text{if there exists a representation of } s \text{ with } i, j \text{ adjacent}, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( k \) is the number of bridges in \( s \) and \( b_{ij} \) is the number of bridges crossed on any path from \( i \) to \( j \). The convex hull of all the vectors \( x(s) \) for \( s \) any binary level-1 network with \( n \) leaves and \( k \) nontrivial bridges is the level-1 network polytope \( \text{BME}(n, k) \). It has dimension \( \binom{n}{2} - n \). When \( k = 0 \) it is the Symmetric Travelling Salesman polytope \( \text{STSP}(n) \).

Examples: vertex vectors \( x(s) \).

\[
(4, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 2, 4, 0, 4) \quad (2, 0, 1, 0, 1, 2, 0, 0, 0, 1, 0, 1, 2, 0, 2)
\]

Theorem: Counting vertices.

The number of vertices of \( \text{BME}(n, k) \), and thus the number of binary level-1 networks with \( n \) leaves and \( k \) non-trivial bridges, with \( 0 \leq k \leq n - 3 \) is:

\[
v(n, k) = T(n, k) \binom{n-1}{k} \frac{(n-1)!}{2^k}
\]

where \( T(n, k) \) gives the components of the face vector of the associahedron \( K(n) \), which allows the simpler count:

\[
v(n, k) = \binom{n-3}{k} \binom{n+k-1}{k} \frac{(n+k-1)!}{(2k+2)!}
\]

Example: Duality ...

BME(4, 0) on the left is dual to CSN(4) on the right. BME(4, 1) is dual to BHV(4); these are the restrictions to trees shown by the dashed lines.

Facets: new and old polytopes.

A split facet (a) in \( \text{STSP}(5) = \text{BME}(5, 0) \), a facet (b) in \( \text{BME}(5, 1) \), and a face (c) in \( \text{BME}(5, 2) = \text{BME}(5) \). All three correspond to the same split, pictured at the top.

Definition: New Spaces.

A weighted split network has non-negative real values assigned to each split. The geometric space \( \text{CSN}_n \) of metric circular split networks with \( n \) labeled leaves is defined in [2]. A vector of weights, or edge lengths in the network: \( l_1, \ldots, l_{n(n-3)/2} \) specifies a point in the orthant \([0, \infty)^{n(n-3)/2}\). Orthants are glued along their common boundaries by identifying equivalent networks (with edge lengths of 0). When we restrict to trees we recover the Billera-Holmes-Vogtman space of phylogenetic trees \( \text{BHV}(n) \), [1].

...Theorem: Duality...

There exists a poset injection: \( f : \mathcal{L}(\text{BHV}_n) \rightarrow \mathcal{L}(\text{BME}(n, n-3)^2) \). In particular the \((2n-5)!!\) top-dimensional cells of \( \text{BHV}_n \) map to the \((2n-5)!!\) vertices of \( \text{BME}(n, n-3) \).

The polytope \( \text{STSP}(n) \) has a complex of subfaces which is the dual image of a projection \( f \) from \( \text{CSN}_n \). The \((n-1)!! \) orthants of \( \text{CSN}_n \) map to the vertices of \( \text{STSP}(n) \). See Example to the left.