Split Network Polytopes and Network Spaces

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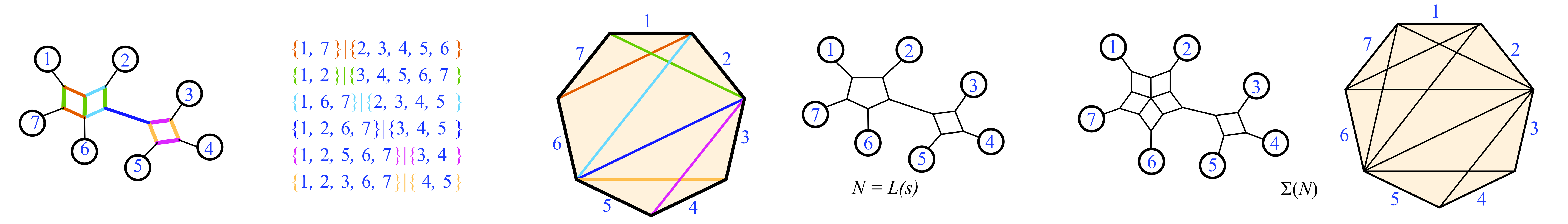
Definitions with examples: Split Networks and Level-1 Phylogenetic Networks.

(1) (2) (3) (4) (5) (6)



Above, we see (1) a *circular split network* s, which represents the collection of *non-trivial splits* $A|B$ in the *split system* (2). The same system is represented by (3) the subdivided polygon. Next there is pictured (4) a *level-1 phylogenetic network* N as defined in [3], which is found by forgetting the internal structure of s via the map L. From any level-1 network we can recover a split network $\Sigma(N)$ which has all the splits corresponding to *minimal cuts* in N as shown in two representations: (5) and (6).

Definition: New Polytopes.

The vector $\mathbf{x}(s)$ is defined to have lexicographically ordered components $x_{ij}(s)$ for each unordered pair of distinct leaves $i, j \in [n]$ as follows:

$$x_{ij}(s) = \begin{cases} 2^{k-b_{ij}} & \text{if there exists a representation of } s; \text{ with } i,j \text{ adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

where k is the number of bridges in s and b_{ij} is the number of bridges crossed on any path from i to j.

The convex hull of all the vectors $\mathbf{x}(s)$ for s any binary level-1 network with n leaves and k nontrivial bridges is the *level-1 network polytope* BME(n,k). It has dimension $\binom{n}{2} - n$. When $k = 0$ it is the Symmetric Travelling Salesman polytope STSP(n).

Examples: vertex vectors $\mathbf{x}(s)$.

(4, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 2, 4, 0, 4) (2, 0, 1, 0, 1, 2, 0, 0, 0, 1, 0, 1, 2, 0, 2)

Theorem: Counting vertices.

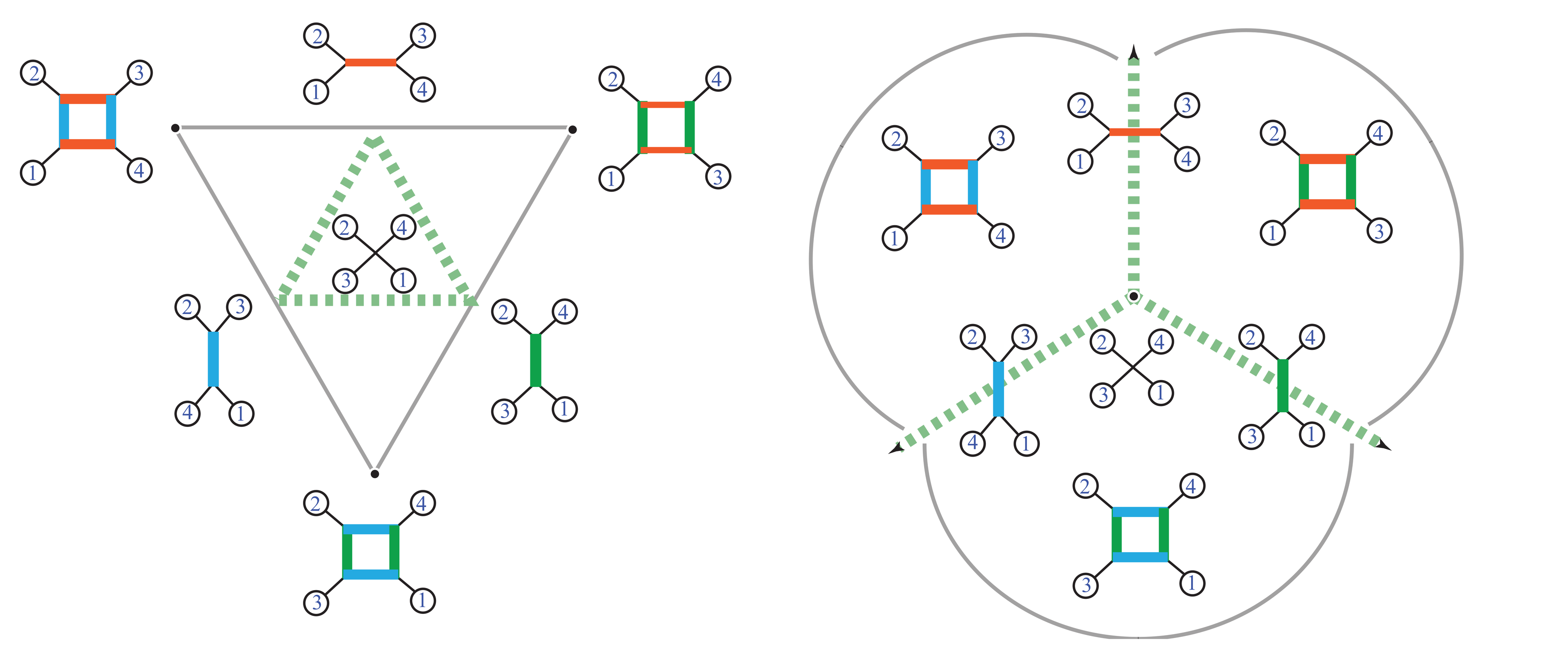
The number of vertices of BME(n,k), and thus the number of binary level-1 networks with n leaves and k non-trivial bridges, with $0 \le k \le n-3$ is:

$$v(n,k) = T(n,k)\frac{(n-1)!}{2^{k+1}}$$

where $T(n,k)$ gives the components of the face vector of the associahedron $K(n)$, which allows the simpler count:

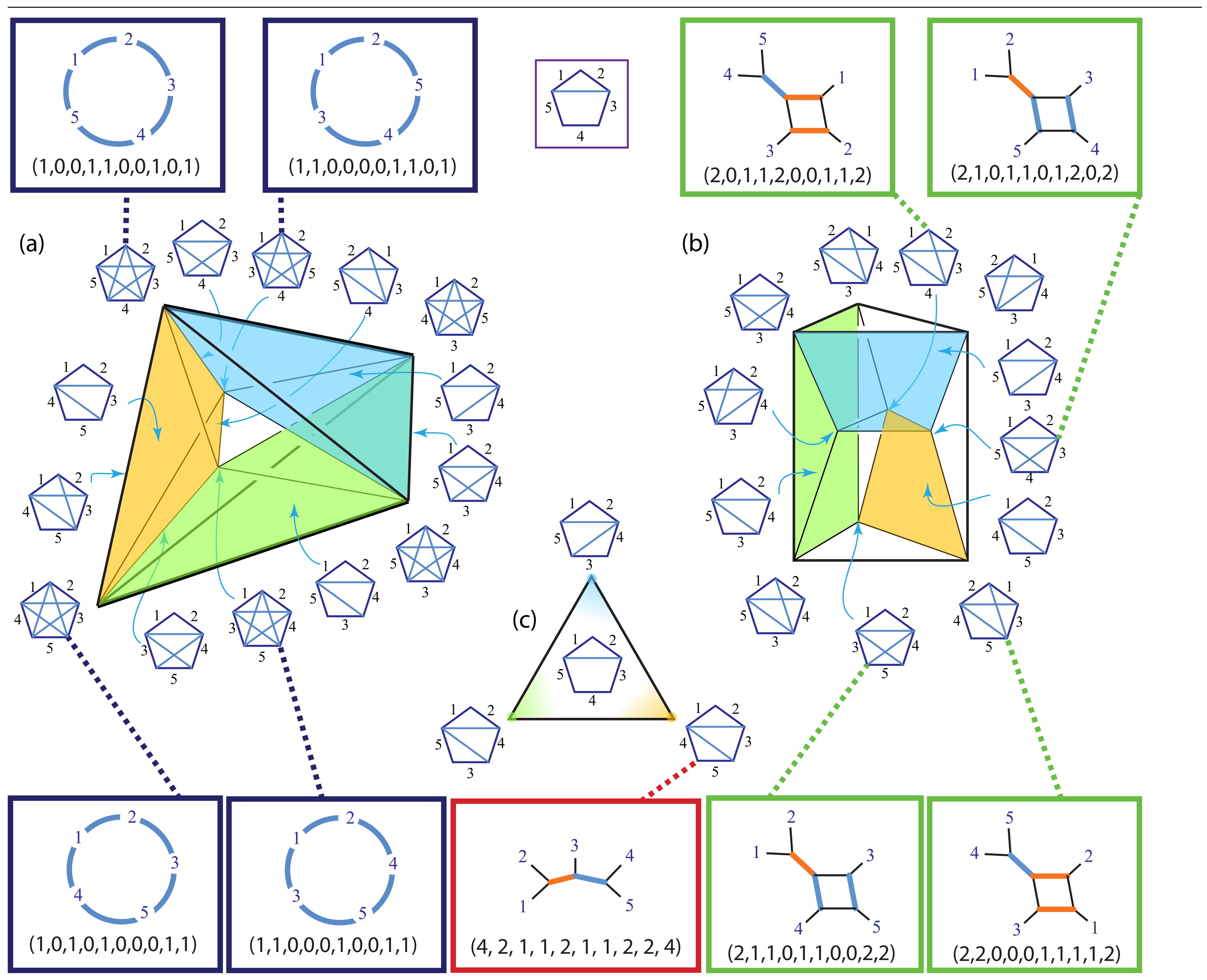
$$v(n,k) = \binom{n-3}{k}\frac{(n+k-1)!}{(2k+2)!!}$$

Example : Duality...



BME(4,0) on the left is dual to CSN(4) on the right. BME(4,1) is dual to BHV(4); these are the restrictions to trees shown by the dashed lines.

Facets: new and old polytopes.



A split facet (a) in STSP(5) = BME(5,0), a facet (b) in BME(5,1), and a face (c) in BME(5,2) = BME(5). All three correspond to the same split, pictured at the top.

Definition: New Spaces.

A *weighted split network* has non-negative real values assigned to each split. The geometric space CSN$_n$ of metric circular split networks with n labeled leaves is defined in [2]. A vector of weights, or edge lengths in the network: $(l_1, \dots, l_{n(n-3)/2})$ specifies a point in the orthant $[0,\infty)^{n(n-3)/2}$. Orthants are glued along their common boundaries by identifying equivalent networks (with edge lengths of 0). When we restrict to trees we recover the Billera-Holmes-Vogtman space of phylogenetic trees BHV(n), [1].

...Theorem: Duality.

There exists a poset injection: $f : \mathcal{L}(\text{BHV}_n) \to \mathcal{L}(\text{BME}(n, n-3)^{\Delta})$. In particular the $(2n-5)!!$ top-dimensional cells of BHV$_n$ map to the $(2n-5)!!$ vertices of BME(n)=BME$(n, n-3)$.

The polytope STSP(n) has a complex of subfaces which is the dual image of a projection f from CSN$_n$. The $(n-1)!/2$ orthants of CSN$_n$ map to the vertices of STSP(n). See Example to the left.

[1] L. Billera, S. Holmes, and K. Vogtmann, *Geometry of the space of phylogenetic trees*, Adv. in Appl. Math. **27** (2001), no. 4, 733–767. MR 1867931

[2] S. Devadoss and S. Petti, *A space of phylogenetic networks*, SIAM Journal on Applied Algebra and Geometry **1** (2017), 683–705.

[3] P. Gambette, K. T. Huber, and G. E. Scholz, *Uprooted phylogenetic networks*, Bulletin of Mathematical Biology **79** (2017), no. 9, 2022–2048.