FPSAC19 poster: CYCLIC QUASI-SYMMETRIC FUNCTIONS

Ron M. Adin* (Bar-Ilan U), Ira M. Gessel (Brandeis U), Vic Reiner (U of Minnesota), Yuval Roichman (Bar-Ilan U)

radin@math.biu.ac.il gessel@brandeis.edu reiner@umn.edu yuvalr@math.biu.ac.il

Abstract

The ring cQSym of cyclic quasi-symmetric functions, and a distinguished subring cQSym^-, are introduced. They are intermediate between the rings of symmetric and quasi-symmetric functions. They have natural bases consisting of fundamental cyclic quasi-symmetric functions, which arise as toric P-partition enumerators for toric posets P with a total cyclic order. The structure constants are determined by cyclic shuffles of permutations. For every non-hook shape λ, the coefficients in the expansion of the Schur function s_λ in this basis are nonnegative, and closely related to cyclic descents. The theory has applications to enumeration (of cyclic shuffles and SYT) by cyclic descents.

Sym and QSym

Recall: a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree is symmetric if for any $t \ge 1$, any two sequences $i_1, \ldots, i_t$ and $j_1, \ldots, j_t$ of distinct positive integers (indices), and any sequence $m_1, \ldots, m_t$ of positive integers (exponents), the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{j_1}^{m_1} \cdots x_{j_t}^{m_t}$ in f are equal; and f is quasi-symmetric if for any $t \ge 1$, any two increasing sequences $i_1 < \cdots < i_t$ and $j_1 < \cdots < j_t$ of positive integers, and any sequence $m_1, \ldots, m_t$ of positive integers, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{j_1}^{m_1} \cdots x_{j_t}^{m_t}$ in f are equal.
Let Sym (QSym) denote the ring of symmetric (respectively quasi-symmetric) functions.

Definition of cQSym

A formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree is cyclic quasi-symmetric function if for any $t \ge 1$, any two increasing sequences $i_1 < \cdots < i_t$ and $i'_1 < \cdots < i'_t$ of positive integers, any sequence $m = (m_1, \ldots, m_t)$ of positive integers, and any cyclic shift $m' = (m'_1, \ldots, m'_t)$ of m, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m'_1} \cdots x_{i'_t}^{m'_t}$ in f are equal.
Let cQSym_n be the additive group of all cyclic quasi-symmetric functions which are homogeneous of degree n, and let $\mathrm{cQSym} = \oplus_{n=0}^{\infty} \mathrm{cQSym}_n$. Then $\mathrm{Sym} \subseteq \mathrm{cQSym} \subseteq \mathrm{QSym}$.

Example

$$x_1^4x_2^2x_3 + x_1x_2^4x_3^2 + x_1^2x_2x_3^4 + + x_3^4x_7^2x_8 + x_3x_7^4x_8^2 + x_3^2x_7x_8^4 + \ldots \in \mathrm{cQSym}_7$$

Monotone cyclic words

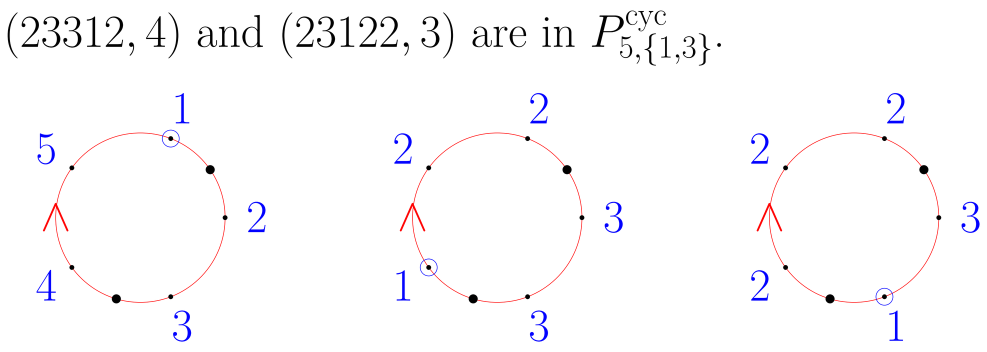
For $n \ge 1$ and a subset $J \subseteq [n]$, let $P_{n,J}^{\mathrm{cyc}}$ be the set of all pairs (w, k) consisting of a word $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ and an index $k \in [n]$ satisfying

(i) $w_k \le w_{k+1} \le \ldots \le w_n \le w_1 \le \ldots \le w_{k-1}$.
(ii) If $j \in J \setminus \{k-1\}$ then $w_j < w_{j+1}$, where indices are computed modulo n.

Note that the word w uniquely determines the index k, unless $w_1 = \ldots = w_n$ and $J = \varnothing$; each of these "constant words" is counted n times (and not just once) in $P_{n,\varnothing}^{\mathrm{cyc}}$.

Example

The pairs $(12345, 1)$, $(23312, 4)$ and $(23122, 3)$ are in $P_{5,\{1,3\}}^{\mathrm{cyc}}$.



Fundamental basis

For any subset $J \subseteq [n]$ define the corresponding fundamental cyclic quasi-symmetric function by

$$F_{n,J}^{\mathrm{cyc}} := \sum_{(w,k) \in P_{n,J}^{\mathrm{cyc}}} x_{w_1} x_{w_2} \cdots x_{w_n}.$$

$F_{n,J}^{\mathrm{cyc}}$ is invariant under cyclic rotations of J. Let $c2^{[n]}$ be the set of equivalence classes (orbits), under cyclic rotations, of subsets $\varnothing \subseteq J \subseteq [n]$. For an orbit $A \in c2^{[n]}$ define

$$F_{n,A}^{\mathrm{cyc}} := F_{n,J}^{\mathrm{cyc}} \quad (\forall J \in A).$$

The corresponding normalized fundamental cyclic quasi-symmetric function is

$$\hat{F}_{n,A}^{\mathrm{cyc}} := \frac{1}{n} \sum_{J \in A} F_{n,J}^{\mathrm{cyc}}.$$

Dependence and basis

Denoting $r(A) = |J|$ (for any $J \in A$), there is a unique linear dependence:

$$\sum_{A \in c2^{[n]}} (-1)^{r(A)} \hat{F}_{n,A} = 0,$$

The set $\left\{\hat{F}_{n,A}^{\mathrm{cyc}} \,:\, A \in c2^{[n]} \setminus \{[\varnothing]\}\right\}$ is a $\mathbb{Z}$-basis for cQSym_n.

Definition of cQSym^-

For many combinatorial applications it is natural to consider a certain subring cQSym_n^- of cQSym_n. Define

$$\mathrm{cQSym}_n^- := \mathrm{span}_{\mathbb{Z}} \left\{\hat{F}_{n,A}^{\mathrm{cyc}} \,:\, A \in c2^{[n]} \setminus \{[\varnothing], [[n]]\}\right\} \quad (n > 1),$$

as well as $\mathrm{cQSym}_1^- := \mathrm{span}_{\mathbb{Z}}\left\{\hat{F}_{1,[[1]]}^{\mathrm{cyc}}\right\}$, $\mathrm{cQSym}_0^- := \mathbb{Z}$, and $\mathrm{cQSym}^- := \oplus_{n \ge 0} \mathrm{cQSym}_n^-$.

Descents and cyclic descents

The descent set of a permutation $\pi = (\pi_1, \ldots, \pi_n) \in \mathfrak{S}_n$ is

$$\mathrm{Des}(\pi) := \{1 \le i \le n-1 \,:\, \pi_i > \pi_{i+1}\} \subseteq [n-1].$$

The cyclic descent set is defined by

$$\mathrm{cDes}(\pi) := \{1 \le i \le n \,:\, \pi_i > \pi_{i+1}\} \subseteq [n],$$

where $\pi_{n+1} := \pi_1$. The descent and cyclic descent numbers are $\mathrm{des}(\pi) := |\mathrm{Des}(\pi)|$ and $\mathrm{cdes}(\pi) := |\mathrm{cDes}(\pi)|$. E.g., for $\pi = 23154$:

$$\mathrm{Des}(\pi) = \{2, 4\}, \quad \mathrm{cDes}(\pi) = \{2, 4, 5\}.$$

Cyclic shuffles

$$[345] = \{345, 453, 534\} \qquad [12] \sqcup\!\sqcup_{\mathrm{cyc}} [345] = \{[12345], [13245], [13425], [13452], [12453], [14253], [14523], [14532], [12534], [15234], [15324], [15342]\}$$

The following result is a cyclic analogue of Stanley's shuffle theorem.

Thm: cQSym product and cyclic shuffles

Let $C = A \sqcup B$ be a disjoint union of two finite sets of integers. For each $u \in \mathfrak{S}_A$ and $v \in \mathfrak{S}_B$,

$$F_{|A|,[\mathrm{cDes}(u)]}^{\mathrm{cyc}} \cdot F_{|B|,[\mathrm{cDes}(v)]}^{\mathrm{cyc}} = \sum_{[w] \in [u] \sqcup\!\sqcup_{\mathrm{cyc}} [v]} F_{|C|,[\mathrm{cDes}(w)]}^{\mathrm{cyc}}.$$

Thm: Counting cyclic shuffles

Let A and B be disjoint sets of integers, $|A| = m$ and $|B| = n$. For each $u \in \mathfrak{S}_A$ and $v \in \mathfrak{S}_B$, if $\mathrm{des}(u) = i$ and $\mathrm{des}(v) = j$ then the number of shuffles of u and v with descent number k is

$$\binom{m+j-i}{k-i}\binom{n+i-j}{k-j};$$

and if $\mathrm{cdes}(u) = i$ and $\mathrm{cdes}(v) = j$ then the number of cyclic shuffles of $[u]$ and $[v]$ with cyclic descent number k is

$$\frac{k(m-i)(n-j) + (m+n-k)ij}{(m+j-i)(n+i-j)} \binom{m+j-i}{k-i}\binom{n+i-j}{k-j}.$$

A tropical tool in the proof

Let $\mathbb{Z}[[q]]_\odot$ be the ring of formal power series in q, with product $q^i \odot q^j := q^{\max(i,j)}$. Use the ring homomorphism $\Psi : \mathrm{cQSym} \to \mathbb{Z}[[q]]_\odot$ defined by

$$\Psi(x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}) := q^{i_t} \quad (i_1 < \ldots < i_t).$$

Cyclic descents for SYT

Cyclic descents can be defined for standard Young tableaux (SYT) of any skew shape λ/μ which is not a connected ribbon [Adin-Reiner-Roichman, Huang].

Thm: Schur function expansion

For every skew shape λ/μ of size n, which is not a connected ribbon, and for any cyclic extension (cDes, p) of Des on $\mathrm{SYT}(\lambda/\mu)$,

$$s_{\lambda/\mu} = \sum_{A \in c2_{0,n}^{[n]}} m^{\mathrm{cyc}}(A)\, \hat{F}_{n,A}^{\mathrm{cyc}}$$

where, for $J \in A \in c2_{0,n}^{[n]} = c2^{[n]} \setminus \{[\varnothing], [[n]]\}$,

$$m^{\mathrm{cyc}}(A) := m^{\mathrm{cyc}}(J) = |\{T \in \mathrm{SYT}(\lambda/\mu) \,:\, \mathrm{cDes}(T) = J\}|$$

is a nonnegative integer, equal [Postnikov 2005] to a certain Gromov-Witten invariant.

Thm: Coideals and a cyclic descent module

cQSym_n and cQSym_n^- are right coideals of QSym_n with respect to the internal coproduct:

$$\Delta_n(\mathrm{cQSym}_n) \subseteq \mathrm{cQSym}_n \otimes \mathrm{QSym}_n \quad \text{and} \quad \Delta_n(\mathrm{cQSym}_n^-) \subseteq \mathrm{cQSym}_n^- \otimes \mathrm{QSym}_n.$$

The structure constants for cQSym_n^- are nonnegative integers.
For $n \le 5$, cQSym_n and cQSym_n^- are also left coideals of QSym_n, thus coalgebras.

By duality, cyclic descent classes form a left module for Solomon's descent algebra.