Stanley symmetric functions for affine involutions

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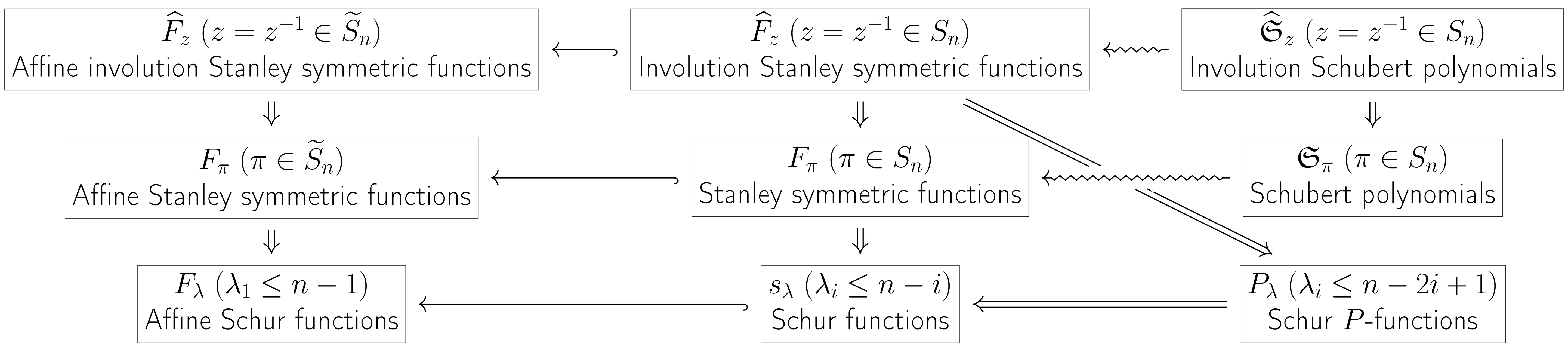
Families of polynomials and symmetric functions



Affine permutations

- The *symmetric group* S_n is the group of bijections $\pi : [n] \to [n] := \{1, 2, \dots, n\}$.
- The *affine symmetric group* $\widetilde{S}_n$ is the group of bijections $\pi : \mathbb{Z} \to \mathbb{Z}$ satisfying
$$\pi(i+n) = \pi(i) + n \quad \text{and} \quad \pi(1) + \pi(2) + \dots + \pi(n) = 1 + 2 + \dots + n.$$
The finite subgroup of elements $\pi \in \widetilde{S}_n$ with $\pi([n]) = [n]$ may be identified with S_n.
- Let $s_i \in \widetilde{S}_n$ be permutation interchanging $i \leftrightarrow i+1$, fixing all $j \notin \{i, i+1\} + n\mathbb{Z}$. Then $s_i = s_{i+n}$ and $\widetilde{S}_n = \langle s_1, \dots, s_n \rangle$ is a Coxeter group with *length function* ℓ.

Affine Stanley symmetric functions

- A permutation $\pi \in \widetilde{S}_n$ is *cyclically decreasing* if it has a reduced expression $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}$ with $i_j + 1 \notin i_k + n\mathbb{Z}$ for all $1 \leq j < k \leq l$.
- Lam (2006): The *(affine) Stanley symmetric function* of $\pi \in \widetilde{S}_n$ is
$$F_\pi = \sum_{\pi = \pi^1 \pi^2 \cdots} x_1^{\ell(\pi^1)} x_2^{\ell(\pi^2)} \cdots \in \mathbb{Z}[[x_1, x_2, \dots]]$$
where the sum is over all factorizations $\pi = \pi^1 \pi^2 \cdots$ of π into countably many cyclically decreasing factors $\pi^i \in \widetilde{S}_n$ with $\ell(\pi) = \ell(\pi^1) + \ell(\pi^2) + \dots$.
- The power series F_π are analogues of Schubert polynomials for affine Grassmannian. When $\pi \in S_n \subsetneq \widetilde{S}_n$, the F_π are stable limits of Schubert polynomials for flag variety.
- The functions F_π with $\mathrm{Des}_L(\pi) := \{i \in \mathbb{Z} : \pi^{-1}(i+1) < \pi^{-1}(i)\} = n\mathbb{Z}$ are called *affine Schur functions*. These include each s_λ with $\lambda \subseteq (n-1, \dots, 2, 1)$.

Theorem [Edelman and Greene (1987); Lam (2008)]

The affine Schur functions are a basis for the free abelian group
$$\Lambda^{(n)} := \mathbb{Z}\text{-span}\{m_\lambda : \lambda_1 < n\} = \mathbb{Z}\text{-span}\{F_\pi : \pi \in \widetilde{S}_n\}.$$
If $\pi \in S_n \subsetneq \widetilde{S}_n$ then F_π is symmetric and *Schur positive*. If $\pi \in \widetilde{S}_n - S_n$ then F_π is not necessarily Schur positive, but is always symmetric and *affine Schur positive*.

Affine transition formula

- For $i < j \not\equiv i \pmod n$, let $t_{ij} \in \widetilde{S}_n$ interchange $i \leftrightarrow j$, fixing all $k \notin \{i, j\} + n\mathbb{Z}$. The *Bruhat order* on $\widetilde{S}_n$ is the transitive closure of the relation with $\pi \lessdot \sigma$ if $\ell(\sigma) = \ell(\pi) + 1$ and $\sigma = \pi t_{ij}$ for some $i < j \not\equiv i \pmod n$.

Theorem [Lam and Shimozono (2006)]

If $\pi \in \widetilde{S}_n$ and $r \in \mathbb{Z}$ then $\sum_{\sigma \in \Psi_r^-(\pi)} F_\sigma = \sum_{\sigma \in \Psi_r^+(\pi)} F_\sigma$ where
$$\Psi_r^-(\pi) := \left\{ \sigma \in \widetilde{S}_n : \pi \lessdot \sigma = \pi t_{ir} \text{ for some } i < r \text{ with } i \notin r + n\mathbb{Z} \right\},$$
$$\Psi_r^+(\pi) := \left\{ \sigma \in \widetilde{S}_n : \pi \lessdot \sigma = \pi t_{rj} \text{ for some } j > r \text{ with } j \notin r + n\mathbb{Z} \right\}.$$

- Possible (unrealized) application: combinatorial proof of affine Schur positivity. Pawlowski (2017) used the transition formula to construct GL_n-modules whose characters represent cohomology classes of positroid varieties.

Involution Stanley symmetric functions

- The *Demazure product* on $\widetilde{S}_n$ is the associative operation $\circ : \widetilde{S}_n \times \widetilde{S}_n \to \widetilde{S}_n$ with
$$\pi \circ s_i = \pi \text{ if } \pi(i) > \pi(i+1) \quad \text{and} \quad \pi \circ s_i = \pi s_i \text{ if } \pi(i) < \pi(i+1).$$
Let $I_n := \{z \in S_n : z = z^{-1}\} \subsetneq \widetilde{I}_n := \{z \in \widetilde{S}_n : z = z^{-1}\} = \{\pi^{-1} \circ \pi : \pi \in \widetilde{S}_n\}$.
- M., Z. (2018): The *(affine) involution Stanley symmetric function* of $z \in \widetilde{I}_n$ is
$$\widehat{F}_z = \sum_{\pi \in \mathcal{A}(z)} F_\pi \in \mathbb{Z}[[x_1, x_2, \dots]]$$
where $\mathcal{A}(z)$ is the set of $\pi \in \widetilde{S}_n$ of minimal length such that $\pi^{-1} \circ \pi = z$.

For $z \in I_n \subsetneq \widetilde{I}_n$, definition was considered earlier by Hamaker, M., and Pawlowski.
- Conjecture: If $z \in \widetilde{I}_n$ then also $\widehat{F}_z = \sum_{\pi \in \mathcal{A}(z)} F_{\pi^{-1}}$. (This holds for $z \in I_n \subsetneq \widetilde{I}_n$.)

- Each $\widehat{F}_z$ is symmetric and affine Schur positive. The functions $\widehat{F}_z$ with
$$\mathrm{Des}_V(z) := \{i \in \mathbb{Z} : z(i+1) < \min\{i, z(i)\}\} = n\mathbb{Z}$$
are affine analogues of *Schur P-functions*. These include each Schur P-function P_λ with $\lambda \subseteq (n-1, n-3, n-5, \dots)$. Results of Hamaker, M., and Pawlowski (2017) show that if $z \in I_n \subsetneq \widetilde{I}_n$ then $\widehat{F}_z$ is an $\mathbb{N}$-linear combination of these functions.
- However, such "affine Schur P-functions" are not a basis for $\mathbb{Z}\text{-span}\{\widehat{F}_z : z \in \widetilde{I}_n\}$. It is an open problem to find a natural basis for this free abelian group. It is also an open problem to find a geometric interpretation of $\widehat{F}_z$, and to relate these to Lam, Schilling, and Shimozono's "type C" affine Stanley symmetric functions.

Involution transition formula

- Let $i < j \not\equiv i \pmod n$ be positive integers.

Theorem [M., Z. (2018)]

There exists a unique map $\tau_{ij}^n : \widetilde{I}_n \to \widetilde{I}_n$ such that for each $z \in \widetilde{I}_n$, either:

- There is some $w \in \mathcal{A}(z)$ such that $w \lessdot w t_{ij} \in \mathcal{A}(\tau_{ij}^n(z))$.
- No $w \in \mathcal{A}(z)$ exists with $w \lessdot w t_{ij} \in \mathcal{A}(v)$ for some $v \in \widetilde{I}_n$, and $\tau_{ij}^n(z) = z$.

The Bruhat order of $\widetilde{S}_n$ restricted to $\widetilde{I}_n$ is the transitive closure of the relation with $y \lessdot z$ if $z = \tau_{ij}^n(y) \neq y$ for some $i < j \not\equiv i \pmod n$.

- This theorem is effective: the maps τ_{ij}^n have a complicated but explicit definition.

Theorem [M., Z. (2018)]

If $y \in \widetilde{I}_n$ and $p \leq q = y(p)$ then $\sum_{z \in \Phi_p^-(y)} \widehat{F}_z = \sum_{z \in \Phi_q^+(y)} \widehat{F}_z$ where
$$\Phi_p^-(y) := \left\{ z \in \widetilde{I}_n : y \lessdot_I z = \tau_{ip}^n(y) \text{ for some } i < p \text{ with } i \notin \{p, q\} + n\mathbb{Z} \right\}$$
$$\Phi_q^+(y) := \left\{ z \in \widetilde{I}_n : y \lessdot_I z = \tau_{qj}^n(y) \text{ for some } j > q \text{ with } j \notin \{p, q\} + n\mathbb{Z} \right\}.$$

- It may be possible to use this to expand $\widehat{F}_z$ into summands of some canonical form.

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