Delta conjectures

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Introduction

The *shuffle conjecture* [10], now a theorem by Carlsson and Mellit in [2], gives a combinatorial interpretation of the symmetric function ∇e_n (which is also the Frobenius characteristic of the module of *diagonal harmonics* [12, 11]) in terms of *labelled Dyck paths*. In [13] the authors propose an interpretation of $\nabla\omega(p_n)$ in terms of *labelled square paths*, known as *square conjecture*, now a theorem by Sergel in [15]. In [9], the authors propose an interpretation of $\Delta_{h_m}\Delta'_{e_{n-k-1}}e_n$ in terms of *(partially) labelled decorated Dyck paths*, known as *(generalized) Delta conjecture*. Zabrocki then conjectures a module for the Delta conjecture (i.e. the case $m = 0$) [16]. In our work [4], we merge the two generalizations by suggesting an interpretation of $\frac{[n-k]_t}{[n]_t}\Delta_{h_m}\Delta_{e_{n-k}}\omega(p_n)$ in terms of *partially labelled decorated square paths*. Moreover, we prove several special cases, as the $\langle \cdot, h_d e_{n-d}\rangle$ case of both the generalized Delta [6] and Delta square [4], the $t = 0$ case [5] and the $k = 0$ case [7] of the generalized Delta square, and some consequences.

Symmetric functions

We denote by Λ the algebra over the field $\mathbb{Q}(q,t)$ of symmetric functions in the variables $x_1, x_2, \dots$. We denote by e_n, h_n and p_n the *elementary*, *complete homogeneous* and *power symmetric function* of degree n, respectively. We denote by ω the involution of Λ defined by $\omega(e_n) \coloneqq h_n$ for all n.

Also, for any partition μ, we denote by $s_\mu \in \Lambda$ the corresponding *Schur function*. It is well-known that the symmetric functions $\{s_\mu\}_\mu$ form a basis of Λ. The *Hall scalar product* on Λ, denoted $\langle\, , \,\rangle$, can be defined by stating that the Schur functions are an orthonormal basis.

Let $\widetilde{H}_\mu \in \Lambda$ denote the *(modified) Macdonald polynomial* indexed by the partition μ. As the polynomials $\{\widetilde{H}_\mu\}_\mu$ form a basis of Λ, given a symmetric function $f \in \Lambda$, we can define the *Delta operators* Δ_f and Δ'_f on Λ by setting

$$\Delta_f \widetilde{H}_\mu \coloneqq f[B_\mu(q,t)]\widetilde{H}_\mu \quad \text{and} \quad \Delta'_f \widetilde{H}_\mu \coloneqq f[B_\mu(q,t)-1]\widetilde{H}_\mu, \quad \text{for all } \mu,$$

where $B_\mu(q,t) = \sum_{c\in\mu} q^{a'_\mu(c)}t^{l'_\mu(c)}$ ($a'_\mu(c)$ and $l'_\mu(c)$ are the coarm and coleg of c in μ, respectively) and the square brackets denote the *plethystic* substitution: if $X = x_1 + x_2 + \dots$ is a sum of monomials, then $f[X] \coloneqq f(x_1, x_2, \dots)$.

Combinatorial definitions

Definition: A *partially labelled decorated square path ending east* is a square lattice path ending east whose vertical steps are labelled with non-negative integers such that the labels are strictly increasing along columns, there is at least one label $\neq 0$ labelling a vertical step starting from the base diagonal, and if the path starts with a vertical step, this first step's label is $\neq 0$. A *rise* of a square path, is a vertical step preceded by another vertical step. Decorate some of them with a $*$.

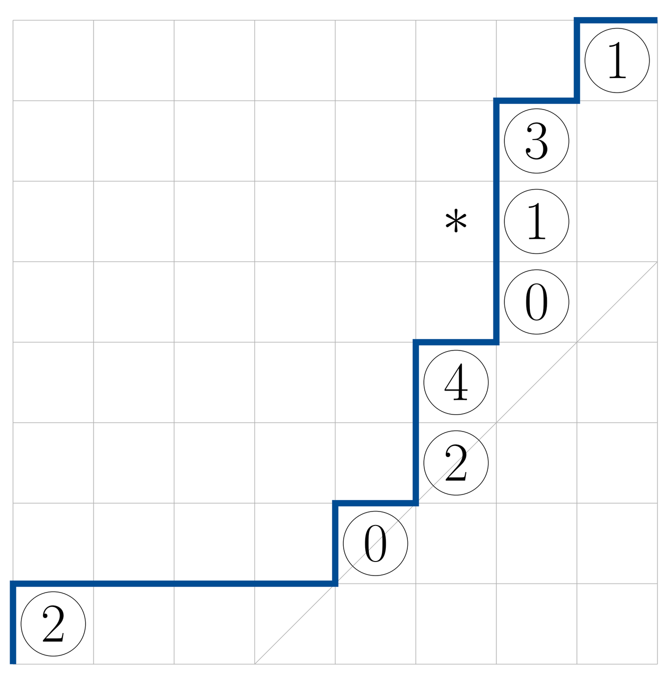

Figure 1. Example of an element in $\mathsf{LSQ}(2,6)^{*1}$.

The set of such paths with m zero and n nonzero labels and k decorated rises is denoted by $\mathsf{LSQ}(m,n)^{*k}$. The subset of the Dyck paths is denoted by $\mathsf{LD}(m,n)^{*k}$.

Definition: the *area* of a path $\mathsf{LSQ}(m,n)^{*k}$ equals the number of whole squares that lie between the path and the lowest diagonal that the path touches and that are not contained in a row containing a rise.

Definition: The dinv of a path $P \in \mathsf{LSQ}(m,n)^{*k}$ is the total number of

- pairs (i,j) with $i < j$ such that the i-th and j-th vertical steps of P lie in the same diagonal and the i-th label $<$ the j-th label of P (starting from the bottom).
- pairs (i,j) with $i < j$ such that the i-th vertical step of P lies one diagonal above the j-th vertical step of P and the i-th label $>$ the j-th label of P.
- labels $\neq 0$ that lie under the line $x = y$.

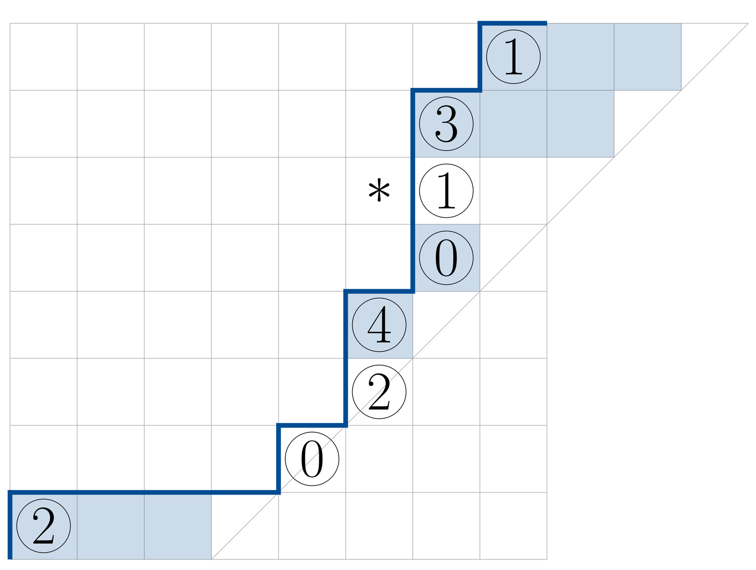

Figure 2. The area of the path is 11.

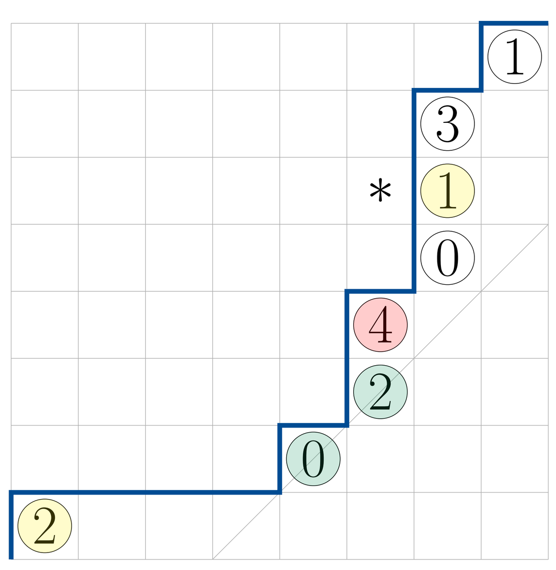

Figure 3. The dinv of the path is 6.

Definition: We define for each $P \in \mathsf{LSQ}(m,n)^{*k}$ a monomial in the variables $x_1, x_2, \dots$: we set $x^P \coloneqq \prod_{i=1}^{n+m} x_{l_i(P)}$ where $l_i(P)$ is the label of the i-th vertical step of P (the first being at the bottom), where we set $x_0 = 1$.

Generalized Delta conjecture [9]

For $m, n, k \in \mathbb{N}$, $m \geq 0$ and $n > k \geq 0$,

$$\Delta_{h_m}\Delta'_{e_{n-k-1}}e_n = \sum_{D\in\mathsf{LD}(m,n)^{*k}} q^{\mathsf{dinv}(D)}t^{\mathsf{area}(D)}x^D$$

History and state of the art

- $m = k = 0$ (*Shuffle conjecture*): Carlsson, Mellit 2015
- $m = 0$ and $q = 0$: Garsia, Haglund, Remmel, Yoo 2017
- $m = 0$ and $q = 1$: Romero 2017
- $m = 0$ and $\langle \cdot, h_{n-d}h_d\rangle$; $m = 0$ and $\langle \cdot, e_{n-d}h_d\rangle$: D-VW 2017
- $\langle \cdot, e_{n-d}h_d\rangle$: D-I-VW 2018
- $t = 0$ or $q = 0$: D-I-VW 2018
- $k = 0$: D-I-VW 2019

Generalized Delta square conjecture [4]

For $m, n, k \in \mathbb{N}$, $m \geq 0$ and $n > k \geq 0$,

$$\frac{[n-k]_t}{[n]_t}\Delta_{h_m}\Delta_{e_{n-k}}\omega(p_n) = \sum_{D\in\mathsf{LSQ}(m,n)^{*k}} q^{\mathsf{dinv}(D)}t^{\mathsf{area}(D)}x^D$$

History and state of the art

- $m = k = 0$ and $\langle \cdot, e_n\rangle$ (q,t-square): Can, Loehr 2006
- $m = k = 0$: Sergel 2016
- $\langle \cdot, e_{n-d}h_d\rangle$: D-I-VW 2018
- $q = 0$: D-I-VW 2018
- $k = 0$: D-I-VW 2019

Towards a proof of the Delta conjectures

The proof of the Shuffle conjecture by Carlsson and Mellit relies on a refinement called the *compositional shuffle conjecture*, in this fashion now a theorem. This breakthrough permitted Sergel to deduce the proof of the Square conjecture. We very recently found an analogous refinement of the Delta conjecture [7], which might lead to its proof. From here on it might be possible to deduce a proof of the generalized Delta square conjecture.

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