Partitions inside a rectangle: tight asymptotics

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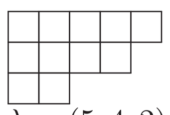
Partitions Asymptotics

Integer partitions λ of n:
$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots), |\lambda| = \lambda_1 + \cdots = n$



$\lambda = (5,4,2) \vdash 11$

$N_n := \#\{\lambda \vdash n\}$

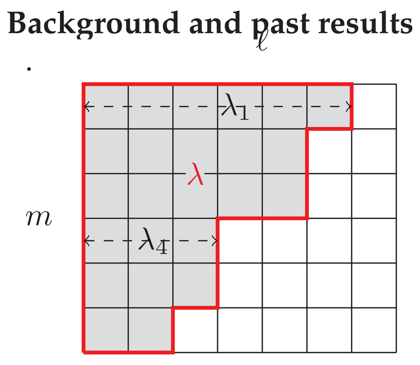
$$\sum_{n=0}^{\infty} N_n q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}\,.$$

Hardy–Ramanujan formula [1]:

$$N_n = \#\{\lambda \vdash n\} \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Partitions inside a rectangle

Background and past results



$N_n(\ell,m) :=$
$\#\{\lambda \vdash n : \lambda_1 \leq \ell, \ell(\lambda) \leq m\}$

$$\sum_{n=0}^{\ell m} N_n(\ell,m) q^n = \binom{\ell+m}{m}_q = \frac{\prod_{i=1}^{\ell+m}(1-q^i)}{\prod_{i=1}^{\ell}(1-q^i)\prod_{i=1}^{m}(1-q^i)}$$

Theorem 1 (Takács, 1986, [2]). *In the regime where $|n-\ell m/2| = O(\sqrt{\ell m(\ell+m)})$ (close to the the middle), and $\ell = \Theta(m), n = \Theta(m^2)$:*

$$N_n(m,\ell) \sim \frac{2^{m+\ell+2}\sqrt{3}}{\pi(m+\ell)^2} \exp\left(-\frac{(m-\ell)^2}{2(m+\ell)} - \frac{3(4n-2m\ell)^2}{2(m+\ell)^3}\right)$$

Theorem 2 (Sylvester 1878, [3], conjectured by Cayley in 1856). *The numbers $N_n(\ell,m)$ form a symmetric* **unimodal sequence**

$$N_0(\ell,m) \leq N_1(\ell,m) \leq \cdots \leq N_{\lfloor m\ell/2 \rfloor}(\ell,m) \geq \cdots \geq N_{m\ell}(\ell,m)$$

Proofs: representation theory of sl_2 [3], Hard Lefshetz Theorem and Linear Algebra Paradigm (Stanley) [4], Combinatorial (O'Hara).
Only bounds on the difference through relation with the representation theory of the symmetric group: The **Kronecker coefficients** – multiplicities of the irreducible representations (Specht modules $\mathbb{S}_\lambda$ for $\lambda \vdash N$) of the symmetric group S_N in the tensor product of two other S_N irreducible representations via diagonal action:

$$g(\lambda,\mu,\nu) := \dim \mathrm{Hom}(\mathbb{S}_\lambda, \mathbb{S}_\mu \otimes \mathbb{S}_\nu)$$

Theorem 3 (Pak–Panova, 2014, [5]). *The consecutive differences are equal to a Kronecker coefficient of the symmetric group $S_{m\ell}$ and satisfy the bound*

$$g((m\ell-n,n),(m^\ell),(m^\ell)) = N_n(\ell,m) - N_{n-1}(\ell,m) \geq 0.004\frac{2^{\sqrt{s}}}{s^{9/4}},$$

where $n \leq \ell m/2$ and $s = \min\{2n,\ell^2,m^2\}$ (e.g. when $\ell = m$ then $s = 2n$).

Limit shape: The limit shape of an unrestricted partition, i.e. the curve which approximates most Young diagrams of $\lambda \vdash n$, was posed as a problem by Vershik and first answered by Szalay and Turan; later Vershik and Yakubovich described the limit shape for singly restricted partitions. The limit shape for partitions inside a rectangle in the regime $m,\ell = \Theta(\sqrt{n})$ was first described by Petrov, where it is identified with a portion of the curve $e^{-x}+e^{-y}=1$, the limit shape of unrestricted partitions. Fluctuations have also been obtained; see [6] for additional historical details and references.

Tight asymptotics of $N_n(m,\ell)$

Regime: $\ell/m \to A$ and $n/m^2 \to B$ for any fixed $A > B > 0$. "Asymptotic formula" we mean: $N_n(\ell,m) = \text{FORMULA}(1+o(1))$, denoted with $\sim$. By symmetry $N_n(\ell,m) = N_{m\ell-n}(\ell,m)$ it suffices to consider only the case $A \geq 2B > 0$.

Given $A \geq 2B > 0$, define c,d as the unique solutions to the equations:

$$A = \int_0^1 \frac{1}{1-e^{-c-td}}dt - 1 = \frac{1}{d}\log\left(\frac{e^{c+d}-1}{e^c-1}\right) - 1\,, \tag{1}$$

$$B = \int_0^1 \frac{t}{1-e^{-c-td}}dt - \frac{1}{2} = \frac{d\log(1-e^{-c-d}) + \mathrm{dilog}\,(1-e^{-c}) - \mathrm{dilog}\,(1-e^{-c-d})}{d^2}\,, \tag{2}$$

$$\mathrm{dilog}\,(x) := \int_1^x \frac{\log t}{1-t}dt = \sum_{k=1}^{\infty}\frac{(1-x)^k}{k^2}\,, \qquad \Delta := \frac{2Be^c(e^d-1)+2A(e^c-1)-1}{d^2(e^{d+c}-1)(e^c-1)} - \frac{A^2}{d^2}$$

Lemma 4. *For any $A > 0$ and $B \in (0, A/2)$ there exist unique $c,d > 0$ satisfying Equations (1) and (2). Moreover, for a fixed A, when B decreases from $A/2$ to 0 then d increases strictly from 0 to ∞ and c decreases strictly from $\log\left(\frac{A+1}{A}\right)$ to 1. When $B > 0$ is fixed and A goes to ∞ then c goes to 0 and d goes to the root of $d^2 = B\left(d\log(1-e^{-d}) - \mathrm{dilog}\,(1-e^{-d})\right)$.*

Theorem 5 (Melczer–Panova–Pemantle, 2018, [6]). *Given m,ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define c,d and Δ as above. Let K be any compact subset of $\{(x,y) : x \geq 2y > 0\}$. As $m \to \infty$ with ℓ and n varying so that (A,B) remains in K,*

$$N_n(\ell,m) \sim \frac{e^{m\left[cA+2dB-\log(1-e^{-c-d})\right]}}{2\pi m^2\sqrt{\Delta\,(1-e^{-c})\,(1-e^{-c-d})}}\,, \tag{3}$$

where c and d vary in a Lipschitz manner with $(A,B) \in K$.

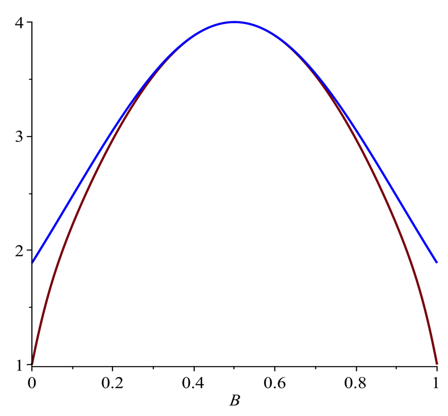
Example 6. *In the special case $B = A/2$, the parameters take on the elementary values*

$$d = 0\,, c = \log\left(\frac{A+1}{A}\right)\,,\ \textit{and}\ \ \Delta = \frac{A^2(A+1)^2}{12}\,.$$

In this case the exponent and leading constant are the limits as $d \to 0$, giving

$$N_{Am^2/2}(Am,m) \sim \frac{\sqrt{3}}{A\pi m^2}\left[\frac{(A+1)^{A+1}}{A^A}\right]^m\,.$$

When $A \to \infty$, so that the restriction on the size of the parts is removed, then $c = 0$, results by Szekeres (circle method), Canfield (recursion), Romik (Fristedt's probability ensemble).



Exponential growth of $N_{Bm^2}(m,m)$ predicted by Takács' formula (blue, above) compared to the actual exponential growth given by Theorem 5 (red, below).

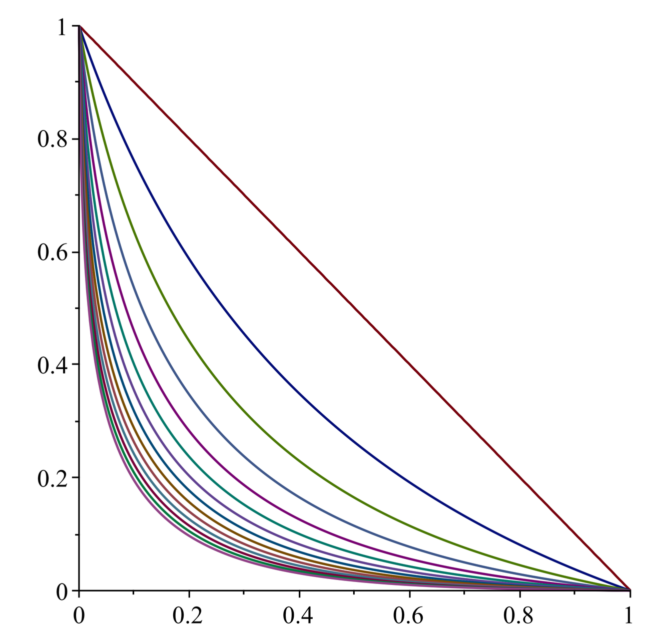
Unimodality: asymptotics of the differences

Theorem 7 (Melczer–Panova–Pemantle, 2018, [6]). *Given m,ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define d as above. Suppose $m,\ell,n \to \infty$, so that (A,B) remains in a compact subset of $\{(x,y) : x \geq 2y > 0\}$ and $m^{-1}|n-\ell m/2| \to \infty$. Then for the consecutive difference of N_n and via Theorem 3 for the Kronecker coefficient we have*

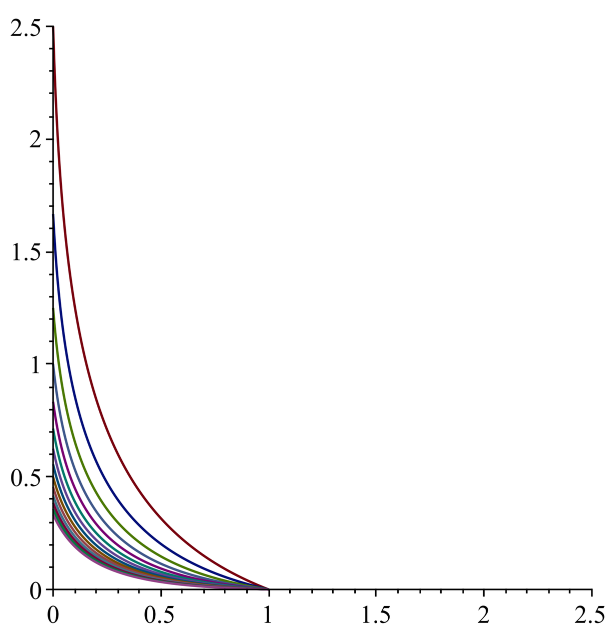
$$g((m\ell-n-1,n+1),m^\ell,m^\ell) = N_{n+1}(\ell,m) - N_n(\ell,m) \sim \frac{d}{m}N_n(\ell,m).$$

Remark. The condition $m^{-1}|n-lm/2| \to \infty$ is equivalent to $m\,|A-B/2| \to \infty$ and also to $d \notin O(m^{-1})$. It is automatically satisfied whenever (A,B) is in a compact subset of $\{(x,y) : x > 2y > 0\}$.

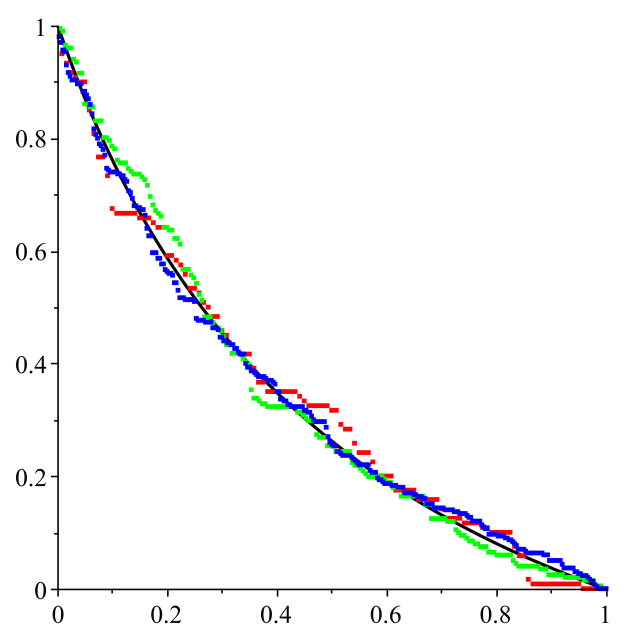
Limit shape



$(A,B) = (1,1/k)$ for $k = 2,\dots,15$



$(A,B) = (5/k,1/k)$ for $k = 2,\dots,15$



Limit curve of $(A,B) = (1,1/3)$ and random partitions of size 120, 201 and 300.

Limit shapes (curves) of scaled partitions as $m \to \infty$: $1 = (1-e^{-c})e^{d(A-y)} + e^{-c}e^{-dx}$ derived from the Random Variables X_i: ↓

$\lambda_i = \ell - (X_0 + X_1 + \cdots + X_{i-1}) \Longrightarrow \mathbb{E}[\lambda_i] = \ell - \sum_{j=0}^{i-1}(1/p_j - 1)$
Set $x = i/m$, approximate the sum by an integral as $m \to \infty$ and get the equation:

$$y := \mathbb{E}\left[\frac{\lambda_i}{m}\right] = A + x - \int_0^x \frac{1}{1-e^{-c-td}}dt = A + x - \frac{1}{d}\ln\left(\frac{e^{xd+c}-1}{e^c-1}\right).$$

Local Central Limit Theorem

Lemma 9 (LCLT, [6]). *Fix $0 < \delta < 1$ and let $p_0,\dots,p_m$ be any real numbers in the interval $[\delta, 1-\delta]$. Let $\{X_j\}$ be independent reduced geometrics with respective parameters $\{p_j\}$, set $S_m := \sum_{j=0}^m X_j$, and $T_m := \sum_{j=0}^m jX_j$. Let M_m be the covariance matrix for (S_m,T_m):*

$$M_m = \begin{pmatrix} \alpha_m m & \beta_m m^2 \\ \beta_m m^2 & \gamma_m m^3 \end{pmatrix},\ Q_m := M_m^{-1},\ \textit{and}\ \Delta_m := m^{-4}\det M_m = \alpha_m\gamma_m - \beta_m^2.$$

Let the means be $\mathbb{E}S_m = \mu_m$ and $\mathbb{E}T_m = \nu_m$ and set $p_m(a,b) := \mathbb{P}((S_m,T_m) = (a,b))$. Then the **probability S_m, T_m concentrate around their means** *satisfies*

$$\sup_{a,b\in\mathbb{Z}} m^2\left|p_m(a,b) - \frac{1}{2\pi(\det M_m)^{1/2}}e^{-\frac{1}{2}Q_m(a-\mu_m,b-\nu_m)}\right| \to 0 \tag{8}$$

as $m \to \infty$, uniformly in the parameters $\{p_j\}$ in the allowed range. If the sequence (a_m,b_m) satisfies $Q_m(a_m-\mu_m, b_m-\nu_m) \to 0$ then

$$p_m(a_m,b_m) = \frac{1}{2\pi\sqrt{\Delta_m}m^2}\left(1 + O\left(m^{-3/2}\right)\right)\,,$$

where for any matrix R we denote $R(s,t) := [s\,,\ t]\,R\,[s\,,\ t]^T$.

Corollary 10 (LCLT consecutive differences, [6]). *Let $\mathcal{N}(a,b) := \frac{1}{2\pi(\det M)^{1/2}}e^{-\frac{1}{2}Q(a-\mu,b-\nu)}$ be the normal approximation in Equation (8).*

$$\sup_{a,b\in\mathbb{Z}}\left|p(a,b+1) - p(a,b) - \left(\mathcal{N}(a,b+1) - \mathcal{N}(a,b)\right)\right| = O(m^{-4}).$$

Proof ideas: Tight approximations and bounds in different regions using characteristic functions.

The continuous Theorem 5

Theorem 5 follows from the discretized Theorem 8 after analyzing c_m, d_m (in particular showing their existence and uniqueness), and their relation to their continuous analogues c and d defined as the solutions of the integral equations (1) and (2) as the Riemann approximations of the summation equations (6). Asymptotics follow from careful analysis of the error bounds.

References

(see [6] for full list)

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[5] Igor Pak and Greta Panova. Bounds on certain classes of Kronecker and q-binomial coefficients. *J. Combin. Theory Ser. A*, 147:1–17, 2017.

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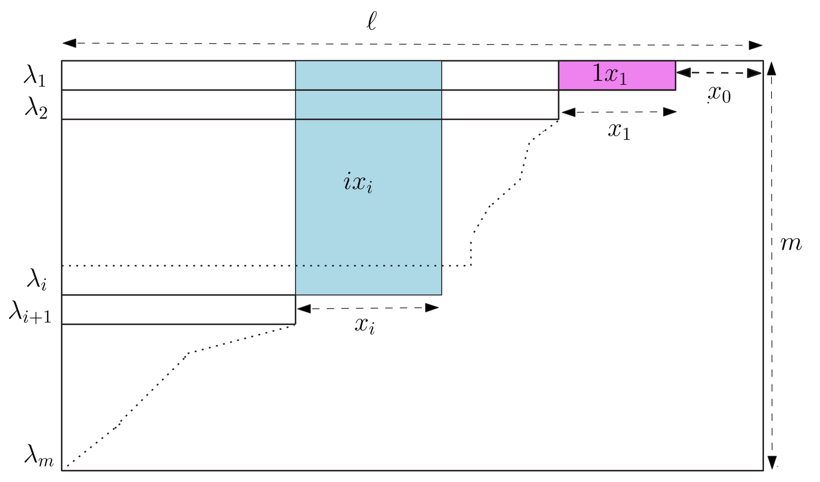
Probabilistic approach

Defining $\lambda_0 := \ell$ and $\lambda_{m+1} := 0$, the *gaps* $x_i := \lambda_i - \lambda_{i+1}$ satisfy

$$\sum_{i=0}^m x_i = \ell\,; \qquad \sum_{i=0}^m ix_i = n\,. \tag{4}$$



The total area n of a partition is composed of rectangles of area jx_j.

Reduced geometric distribution with parameter p: random variable X with $\mathbb{P}(X = k) = p \cdot q^k$ where $q := 1-p$. Let c_m, d_m be solutions to equations (6) below, set

$$q_j := e^{-c_m - jd_m/m}\,, \quad p_j := 1 - q_j\,, \quad L_m := \sum_{j=0}^m \log p_j\,.$$

Tilted geometrics: Let $\mathbb{P}_m$ be a probability law making the random variables $\{X_j : 0 \leq j \leq m\}$ independent reduced geometrics with respective parameters p_j. $\mathbb{E}X_j = 1/p_j - 1 = 1/\left(1-e^{-c_m-d_m j/m}\right) - 1$ and $\mathrm{Var}\,(X_j) = q_j/p_j^2$.
Define **random variables S_m and T_m** by

$$S_m := \sum_{i=0}^m X_i\,; \qquad T_m := \sum_{i=1}^m iX_i\,. \tag{5}$$

$X_j = \lambda_j - \lambda_{j+1}$ for our partitions λ's, so...

λ inside the $m \times \ell$ rectangle $\Longrightarrow \mathbb{E}S_m = \ell$ and $\mathbb{E}T_m = n \Longrightarrow$ (6)

$$\ell = \sum_{j=0}^m \frac{1}{1-e^{-c_m-d_m j/m}} - (m+1)\,, \frac{n}{m} = \sum_{j=0}^m \frac{j/m}{1-e^{-c_m-d_m j/m}} - \frac{m+1}{2}\,,$$

$$\mathrm{Var}\,(S_m) = \sum_{j=0}^m \frac{e^{-c_m-d_m j/m}}{\left(1-e^{-c_m-d_m j/m}\right)^2}\,, \mathrm{Var}\,(T_m) = \sum_{j=0}^m j^2\frac{e^{-c_m-d_m j/m}}{\left(1-e^{-c_m-d_m j/m}\right)^2}\,,$$
$$\mathrm{Cov}\,(S_m,T_m) = \sum_{j=0}^m j\frac{e^{-c_m-d_m j/m}}{\left(1-e^{-c_m-d_m j/m}\right)^2}\,,$$

Theorem 8 (Discretized Theorem 5, [6]). *Let c_m and d_m satisfy equations (6). Define the normalized entries of the covariance matrix:*

$$\alpha_m := m^{-1}\mathrm{Var}\,(S_m)\ ;\ \beta_m := m^{-2}\mathrm{Cov}\,(S_m,T_m)\ ;\ \gamma_m := m^{-3}\mathrm{Var}\,(T_m)\,,$$

which are $O(1)$ as $m \to \infty$.

$$N_n(\ell,m) \sim \frac{1}{2\pi m^2\sqrt{\alpha_m\gamma_m - \beta_m^2}}\exp\left\{m\left(-\frac{L_m}{m} + c_m\frac{\ell}{m} + d_m\frac{n}{m^2}\right)\right\}.$$

Proof outline. The probabilities $\mathbb{P}_m(\mathbf{X} = \mathbf{x})$ depend only on S_m and T_m:

$$\log\mathbb{P}_m(\mathbf{X} = \mathbf{x}) = \sum_{j=0}^m (\log p_j + x_j\log q_j)$$
$$= L_m - \sum_{j=0}^m\left(c_m + j\frac{d_m}{m}\right)x_j = L_m - c_m\left(\sum_{j=0}^m x_j\right) - \frac{d_m}{m}\left(\sum_{j=0}^m jx_j\right).$$

In particular, for any $\mathbf{x}$ satisfying (4),

$$\log\mathbb{P}_m(\mathbf{X} = \mathbf{x}) = L_m - c_m\ell - \frac{d_m}{m}n\,.$$

(i) the vector $\mathbf{X}$ satisfies the identities (4);
$\iff$ (ii) the pair (S_m,T_m) is equal to (ℓ,n);
$\iff$ (iii) the partition $\lambda = (\lambda_1,\dots,\lambda_m)$ defined by $\lambda_j - \lambda_{j+1} = X_j$ for $2 \leq j \leq m-1$, together with $\lambda_1 = \ell - X_0$ and $\lambda_m = X_m$, is a partition of n fitting inside a $m \times \ell$ rectangle.
Setting $p_m(\ell,n) := \mathbb{P}_m\left[(S_m = \mathbb{E}S_m(=\ell), T_m = \mathbb{E}T_m(=n)\right]$ we have

$$N_n(\ell,m) = \frac{p_m(\ell,n)}{\mathbb{P}_m(\mathbf{X} = \mathbf{x})} = p_m(\ell,n)\ \exp\left[m\left(-\frac{L_m}{m} + c_mA + d_mB\right)\right] \tag{7}$$

Now apply the LCLT Lemma 9 with $p_j = 1 - e^{-c_m-d_m j/m}$. □

Consecutive differences

Theorem 7 follows from Equation (7) and Corollary 10. Let

$$L_m(x,y) := \sum_{j=0}^m \log(1-e^{-x-yj/m})\,.$$

$A_m(x,y) := (\partial/\partial x)L_m(x,y), B_m(x,y) := (\partial/\partial y)L_m(x,y)$, then c_m and d_m are the solutions to $A_m(c_m,d_m) = \ell$ and $B(c_m,d_m) = n/m$. Let c'_m, d'_m be the solutions to $A_m(c'_m,d'_m) = \ell$ and $B_m(c'_m,d'_m) = (n+1)/m$, $\Delta x := c'_m - c_m = O(m^{-2})$ and $\Delta y := d'_m - d_m = O(m^{-2})$ coming from analysis of c_m, c, d_m, d. Taylor expansion for $L'_m := L_m(c'_m,d'_m)$ around (c_m,d_m) and the L_m partials gives

$$-L_m(c'_m,d'_m) + (c_m+\Delta x)\ell + (d_m+\Delta y)(n+1)\,m^{-1}$$
$$= -L_m(c_m,d_m) + c_m\ell + d_m(n+1)\,m^{-1} + O(m^{-3}).$$

$N_{n+1}(\ell,m) - N_n(\ell,m)$

$$= p_m(\ell,n)\exp\left[-L_m + c_m\ell + \frac{d_m}{m}n\right]\left[e^{d_m/m} - 1\right] \tag{9}$$

$$+ \left[p_m(\ell,n+1) - p_m(\ell,n)\right]\exp\left[-L_m + c_m\ell + \frac{d_m}{m}(n+1)\right] \tag{10}$$

$$+ p_m(\ell,n+1)\left(e^{-L'_m+c'_m\ell+d'_m(n+1)/m} - e^{-L_m+c_m\ell+d_m(n+1)/m}\right). \tag{11}$$

Bounding each line:
Line (9) $= N_n(\ell,m)\left(\frac{d}{m} + O(m^{-2})\right)$ from Equation (7) since $d_m = d + O(m^{-1})$ for $d \notin O(m^{-1})$ (i.e. $|A-B/2| \notin O(m^{-1})$ as the map $(A,B) \to (c,d)$ is Lipschitz).
Line (10) $= O(m^{-4} \cdot m^2N_n(\ell,m)) = O(m^{-2}N_n(\ell,m))$ since $[p_m(\ell,n+1) - p_m(\ell,n)] \leq |\mathcal{N}(\ell,n+1) - \mathcal{N}(\ell,n)| + O(m^{-4}) = O\left(m^{-2} \cdot \left|1 - e^{\frac{1}{2}Q_m(0,1)}\right|\right) + O(m^{-4}) = O(m^{-4})$, by Corollary 10, where Q_m is the inverse of the covariance matrix of (S_m,T_m).
Line (11) $= p_m(\ell,n+1)\,e^{-L_m+c_m\ell+d_m(n+1)/m}\,\psi_m = N_n(\ell,m)\,\psi_m\,e^{d_m/m} + O(m^{-4}\,e^{d_m/m}\,e^{-L_m+c_m\ell+d_m n/m}\,\psi_m) = O(m^{-3}N_n(\ell,m))$,
since $p_m(\ell,n+1) = p_m(\ell,n) + O(m^{-4})$, where

$$\psi_m := \exp\left[-L'_m + c'_m\ell + \frac{d'_m(n+1)}{m} - \left(-L_m + c_m\ell + \frac{d_m(n+1)}{m}\right)\right] - 1 = O(m^{-3}).$$

$$(9)+(10)+(11) \Longrightarrow N_{n+1}(\ell,m) - N_n(\ell,m) = N_n(\ell,m)\left(\frac{d}{m} + O(m^{-2})\right).\ \square$$