Involution pipe dreams

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Schubert calculus and matrix Schubert varieties

In 1889 Schubert introduced a method for studying problems in enumerative geometry. This method can be formulated in modern terms using intersection theory, e.g. via calculations in the cohomology ring of the flag variety $\mathrm{Fl}(n) := \mathrm{GL}(n,\mathbb{C})/B$, where $B \subseteq \mathrm{GL}(n)$ is a Borel subgroup. The ring $H^*(\mathrm{Fl}(n))$ has a *Schubert basis* $\{[X_w]\}_{w\in S_n}$ determined by *Schubert varieties* X_w, the closures of the left B-orbits on $\mathrm{Fl}(n)$.

The Schubert variety X_w can be described by rank conditions. Let $A_{[i][j]}$ be the upper left $i \times j$ corner of a matrix A, and view $w \in S_n$ as a permutation matrix. Then

$$X_w = \{A \in \mathrm{GL}(n) : \mathrm{rank}(A_{[i][j]}) \leq \mathrm{rank}(w_{[i][j]})\ \forall i,j\}/B.$$

The *orthogonal* and *symplectic* subgroups $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$ of $\mathrm{GL}(n)$ also decompose $\mathrm{Fl}(n)$ into finitely many orbits. Their closures, denoted $\hat{X}_y$ and $\hat{X}_z^{\mathrm{FPF}}$, are indexed by involutions y and fixed-point-free (fpf) involutions z in S_n, respectively.

Let $M(n)$ be the set of $n\times n$ matrices. By analogy, we have the *matrix Schubert variety*

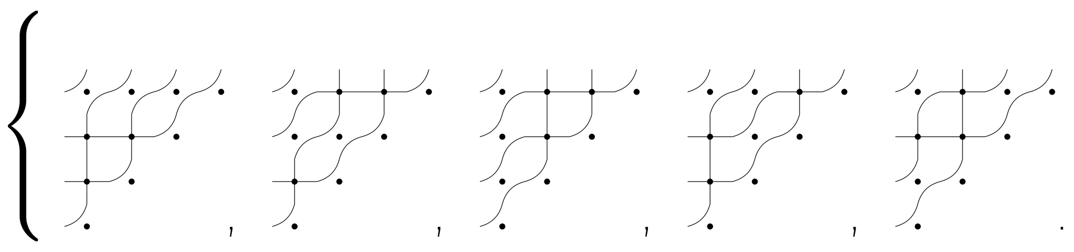
$$MX_w = \{A \in M(n) : \mathrm{rank}(A_{[i][j]}) \leq \mathrm{rank}(w_{[i][j]})\ \forall i,j\}.$$

Using an explicit map $\mathbb{Z}[x_1,\dots,x_n] \twoheadrightarrow H^*(\mathrm{Fl}(n))$ due to Borel, one can represent the class $[X]$ of a subvariety $X \subseteq \mathrm{Fl}(n)$ by a (non-unique) polynomial. By contrast, if V is a representation of the group $T \subseteq \mathrm{GL}(n)$ of diagonal matrices, the equivariant cohomology ring $H_T^*(V)$ is isomorphic to $\mathbb{Z}[x_1,\dots,x_n]$, and we may identify the class $[X] \in H_T^*(V)$ of a T-stable subvariety $X \subseteq V$ with a unique polynomial.

Schubert polynomials and pipedreams

The Schubert polynomials $\mathfrak{S}_w$ are (non-unique) polynomial representatives for $[X_w] \in H^*(\mathrm{Fl}(n))$ and unique representatives for $[MX_w]_T \in H_T(M(n))$. They can be defined as generating functions of combinatorial objects called pipedreams, which we explain by example. The permutation 1432 has five pipedreams:

$$\mathcal{RP}(1432) = \left\{ \quad \right\}$$



The key properties are that the wires of each pipedream are permuted by 1432 as they travel from left to top and that no pair of strands crosses twice. Note that pipedreams are determined by set the of $+$'s contained in $\{(i,j)\in\mathbb{N}^2 : i+j\leq n\}$. Let $\mathcal{RP}(w)$ be the set of pipedreams for w.

Definition: (Billey-Jockusch-Stanley '93)

$$\mathfrak{S}_w = \sum_{D\in\mathcal{RP}(w)} \prod_{(i,j)\in D} x_i.$$

For example, we have

$$\mathfrak{S}_{1432} = x_2^2x_3 + x_1^2x_3 + x_1^2x_2x_1x_2x_3 + x_1x_2^2,$$

where each monomial (from left to right) comes from the corresponding pipedream.

(Skew) Symmetric matrix Schubert varieties

Wyser and Yong have defined polynomial representatives $\hat{\mathfrak{S}}_y$ and $\hat{\mathfrak{S}}_z^{\mathrm{FPF}}$ for $[\hat{X}_y]$ and $[\hat{X}_z^{\mathrm{FPF}}]$ in $H^*(\mathrm{Fl}(n))$. Let SM_n be the set of $n\times n$ symmetric matrices over $\mathbb{C}$, and let SSM_n be the set of $n\times n$ skew symmetric matrices.

Definition: The *symmetric matrix Schubert variety* associated to the involution y is

$$M\hat{X}_y = \{A \in SM_n : \mathrm{rank}(A_{[i][j]}) \leq \mathrm{rank}(y_{[i][j]}) \text{ for } i,j\in[n]\}.$$

The *skew-symmetric matrix Schubert variety* associated to the fpf involution z is

$$M\hat{X}_z^{\mathrm{FPF}} = \{A \in SSM_n : \mathrm{rank}(A_{[i][j]}) \leq \mathrm{rank}(z_{[i][j]}) \text{ for } i,j\in[n]\}$$

Theorem 1

The class $[M\hat{X}_y]$ in $H_T(SM_n)$ equals $2^{\kappa(y)}\hat{\mathfrak{S}}_y$, where $x\in T$ acts by $x.A = xAx$. Similarly, the class $[M\hat{X}_z^{\mathrm{FPF}}]$ in $H_T(SSM_n)$ equals $\hat{\mathfrak{S}}_z^{\mathrm{FPF}}$.

Remarks:

- Theorem 2 generalizes many classical results on degeneracy loci, e.g, Salmon 1862, Segre 1900, Giambelli 1906, Jozefiak-Pragacz 1980
- Work by Fink, Rajchgot and Sullivant relates varieties over SM_n and SSM_n cut out by *north-east* rank conditions to Type B/C Schubert calculus.

A Billey-Jockusch-Stanley formula

A pipedream D is *symmetric* if $(i,j)\in D \Leftrightarrow (j,i)\in D$. It is *almost symmetric* if

- if $(i,j)\in D$ then $(j>i)\in D$ (here $i<j$);
- if $(j,i)\in D$ and $(i,j)\notin D$, the strands crossing at (j,i) also pass through (i,j).

For example, the last two pipedreams of 1432 above are almost symmetric.

Let $\triangle_n = \{(i,j)\in\mathbb{N}^2 : 0 < i \leq j, i+j\leq n\}$ and $\triangle_n^{\neq} = \{(i,j)\in\triangle_n : i\neq j\}$. For y an involution and z a fpf involution in S_n, define

$$\mathcal{IP}(y) = \{D\cap\triangle_n : D\in\mathcal{RP}(y) \text{ and } D \text{ is almost symmetric}\} \text{ and}$$

$$\mathcal{FP}(z) = \{D\cap\triangle_n^{\neq} : D\in\mathcal{RP}(z) \text{ is symmetric with } (i,i)\in D \text{ for } 1\leq i\leq n/2\}.$$

Let $\kappa(y)$ be the number of 2-cycles in y.

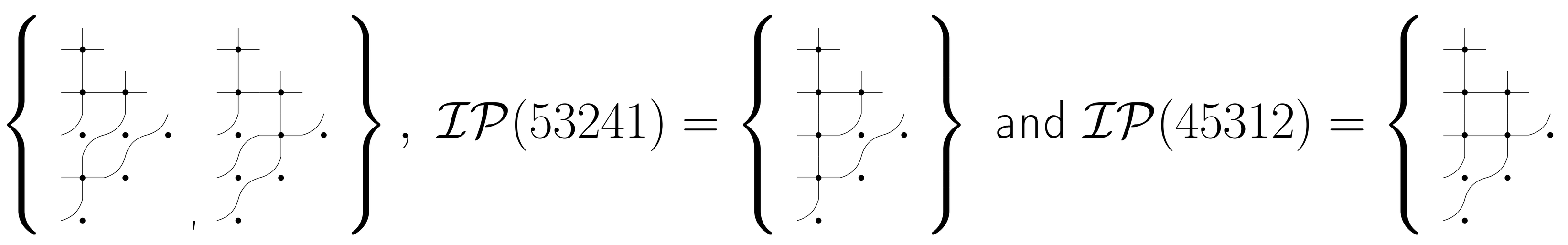
Theorem 2

$$\hat{\mathfrak{S}}_y = 2^{\kappa(y)} \sum_{D\in\mathcal{IP}(y)} \prod_{(i,j)\in D} 2^{-\delta_{ij}}(x_i+x_j) \quad\text{and}\quad \hat{\mathfrak{S}}_z^{\mathrm{FPF}} = \sum_{D\in\mathcal{FP}(z)} \prod_{(i,j)\in D} (x_i+x_j).$$

Example: $\hat{\mathfrak{S}}_{1432} = 2\left((x_2+x_1)(x_3+x_1) + (x_2+x_1)2^{-1}(x_2+x_2)\right).$

Proof of Theorem 2 by Example

For $y = 35142$, we have $\mathcal{IP}(y) = \left\{ \quad \right\}$, $\mathcal{IP}(53241) = \left\{ \quad \right\}$ and $\mathcal{IP}(45312) = \left\{ \quad \right\}$.



Then

$$\hat{\mathfrak{S}}_y = \frac{\hat{\mathfrak{S}}_{53241} + \hat{\mathfrak{S}}_{45312}}{x_1+x_3} = \frac{4x_1x_2(x_1+x_2)(x_1+x_3)((x_1+x_4)+(x_2+x_3))}{x_1+x_3} = 4x_1x_2(x_1+x_2)(x_1+x_4) + 4x_1x_2(x_1+x_2)(x_2+x_3) = 2^{\kappa(y)} \sum_{D\in\mathcal{IP}(y)} \prod_{(i,j)\in D} 2^{-\delta_{ij}}(x_i+x_j).$$

For arbitrary y, the first equality is a recurrence on the $\hat{\mathfrak{S}}_y$'s, the second is a base case for dominant involutions and the final equality requires combinatorial proof.

Connecting the theorems

Our proof of Theorem 1 relies on passing from an $\mathrm{O}(n)$-orbit on $\mathrm{GL}(n)/B$ to a B-orbit on $\mathrm{GL}(n)/\mathrm{O}(n)$ whose closure in SM_n is $M\hat{X}_y$. Geometric considerations then force $\hat{\mathfrak{S}}_y$ to be a sum of monomials in variables (x_i+x_j) as in Theorem 2.

Set-theoretically, $M\hat{X}_y$ is the vanishing locus of an ideal $\hat{I}_y$ generated by minors of a symmetric matrix $\mathcal{X}_n = (x_{ij})_{i\geq j\in[n]}$. For example, $\hat{I}_{2143} = \langle x_{11}, \det\mathcal{X}_3\rangle$. Following Knutson-Miller, we conjecture that $\hat{I}_y$ is prime and that it has a primary decomposition of $\hat{I}_y$ whose top-dimensional components correspond to $\mathcal{IP}(y)$.

Abbreviated Bibliography

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