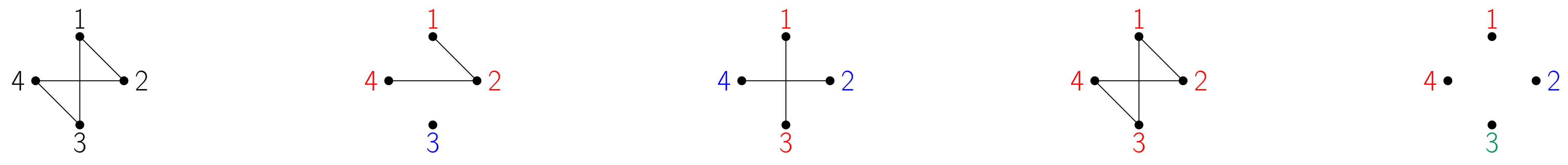
The Noncrossing Bond Poset of a Graph

C. Matthew Farmer *The University of North Carolina at Greensboro* Joshua Hallam† *Loyola Marymount University*

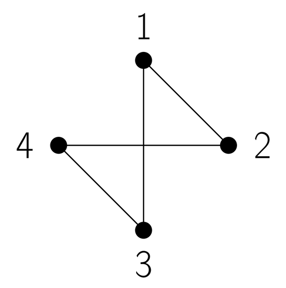
Basic Definitions and Examples

Let G be a graph on $[n]$. A **bond** of G is a partition $\pi(G)$ of $[n]$ such that for each block B of $\pi(G)$ the induced subgraph of G on B is connected. For example, in the graph below, $124/3$, $13/24$, 1234 are all bonds, but $14/2/3$ is not since the induced subgraph on $\{1,4\}$ is not connected.

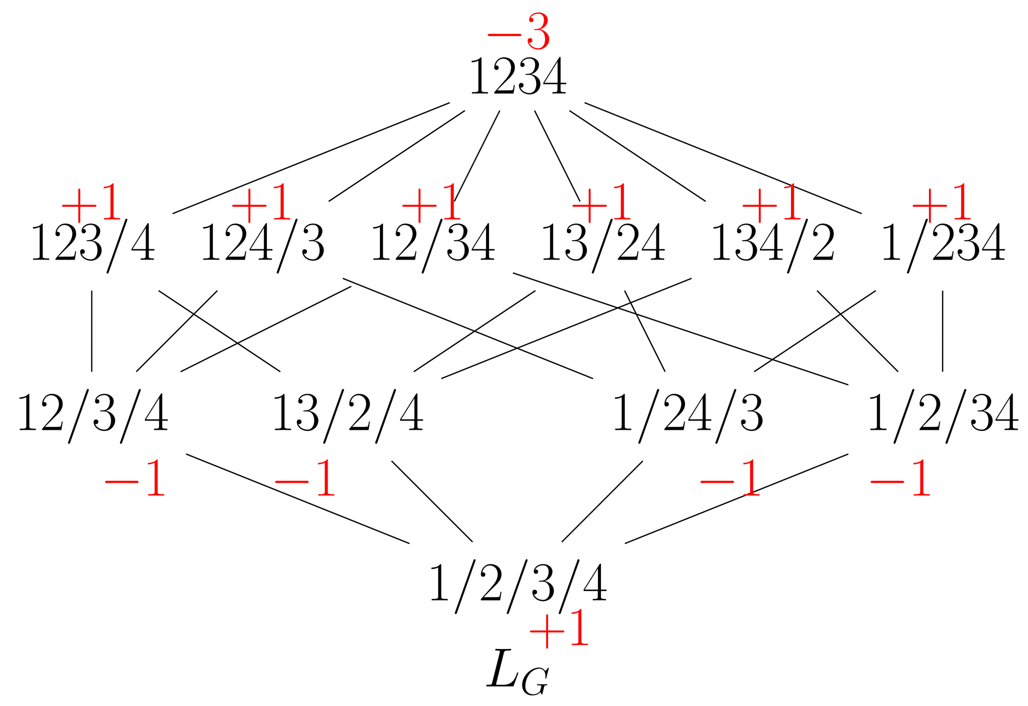


G $\quad 124/3 \quad 13/24 \quad 1234 \quad 14/2/3$

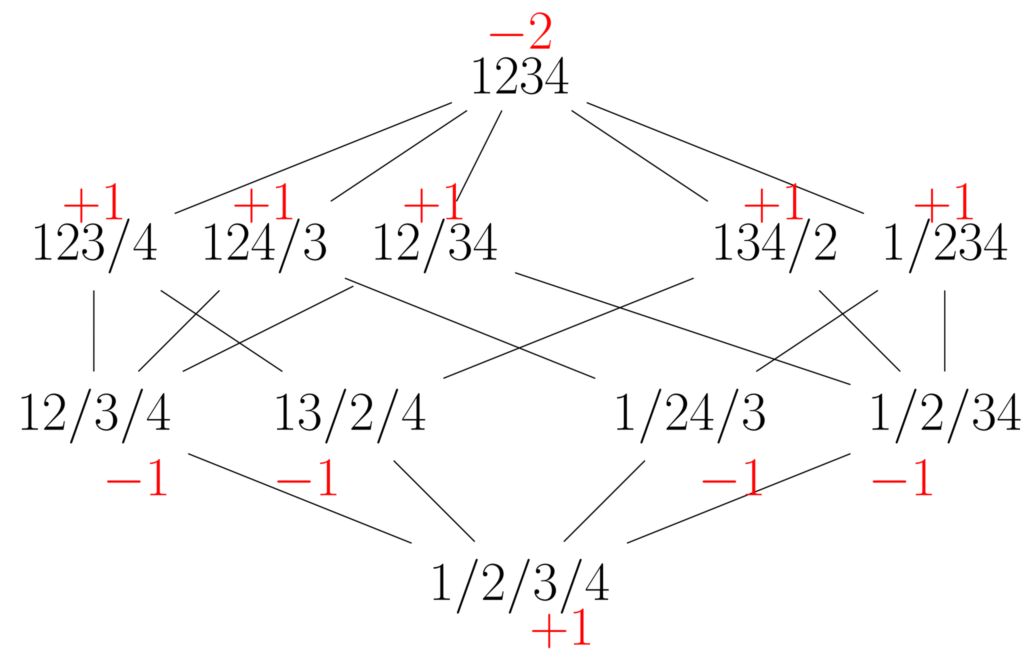
Let $\pi = B_1/B_2/\cdots/B_k$ be a partition of $[n]$. We say π is **crossing** if there are $i \neq j$, $a, c \in B_i$ and $b, d \in B_j$ with $a < b < c < d$, otherwise we say π is **noncrossing**. We say a bond $\pi(G)$ is a **noncrossing bond** if the underlying partition is noncrossing. In the example above, all the bonds are noncrossing except for $13/24$. A noncrossing bond can contain a pair of crossing edges, but only if they are in the same connected component. The **bond lattice**, denoted by L_G, is the poset obtained by ordering the bonds of G by refinement. The **noncrossing bond poset**, denoted by $\mathcal{NC}_G$, is the poset obtained by ordering the noncrossing bonds by refinement.



G



L_G



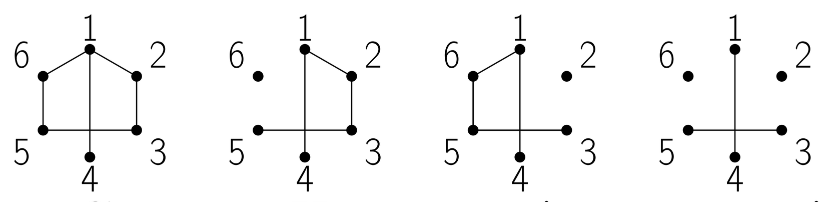
$\mathcal{NC}_G$

Crossing Closed

In general the noncrossing bond poset is not a lattice.

Let G be a graph on $[n]$. We say G is **crossing closed** if whenever $a, b, c, d \in V(G)$ with $a < b < c < d$ and $ac, bd \in E(G)$, there exists a unique minimum (with respect to inclusion) induced connected component containing a, b, c, d.

The graph G below is not crossing closed. There are two minimal induced connected components containing the crossing edges 14 and 35.

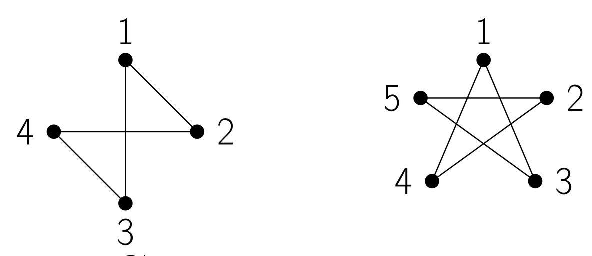


$G \quad H \quad H' \quad H \cap H'$

Proposition $\mathcal{NC}_G$ is a lattice if and only if G is crossing closed.

Let G be a crossing closed graph. Fix an ordering $\trianglelefteq$ on $E(G)$. We say G is **upper crossing closed** with respect to $\trianglelefteq$ if whenever edges ac and bd cross in G, the minimum connected component containing a, b, c, d contains an edge smaller than both ac and bd.

In the example below G is upper crossing closed with respect to the ordering $12 \lhd 13 \lhd 24 \lhd 34$. On the other hand, H is crossing closed, but no ordering of the edges of H can make it upper crossing closed since every edge crosses another edge.



$G \quad H$

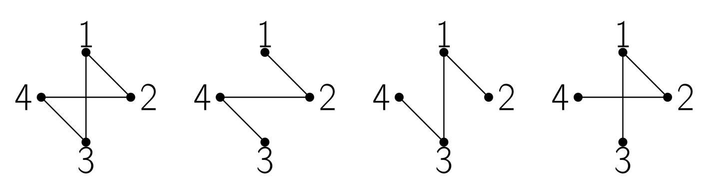
The Möbius Function and Characteristic Polynomial of $\mathcal{NC}_G$

Let P be a graded poset of rank n. The (one-variable) **Möbius function** μ and the characteristic polynomial $\chi(P,t)$ of P are given by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0}, \\ -\sum_{y<x} \mu(y) & \text{otherwise} \end{cases} \quad \text{and} \quad \chi(P,t) = \sum_{x \in P} \mu(x) t^{n-\rho(x)}$$

where $\hat{0}$ is the minimum element of P and $\rho(x)$ is the rank of x. For the example above, $\chi(L_G, t) = t^3 - 4t^2 + 6t - 3$ and $\chi(\mathcal{NC}_G, t) = t^3 - 4t^2 + 5t - 2$. The characteristic polynomial of the bond lattice of G is (up to a factor of t) the chromatic polynomial of G.

The characteristic polynomial of the bond lattice has a combinatorial interpretation in terms of NBC sets. Let G be a graph and let $\trianglelefteq$ be any total ordering on $E(G)$. A **broken circuit** is a collection of edges obtained by removing the smallest edge from a cycle. A **non broken circuit (NBC) set** is a set which does not contain any broken circuits. For example, suppose that we order the edges of G by $12 \lhd 13 \lhd 24 \lhd 34$, the NBC sets with 3 edges are shown below.



$G \quad H_1 \quad H_2 \quad H_3$

Let $nbc_k(G)$ be the number of NBC sets of G with k edges.

***Theorem* (Whitney)** Let G be a graph on $[n]$. Then $\chi(L_G, t) = \sum_{k \geq 0} (-1)^k nbc_k(G) t^{n-k-1}$.

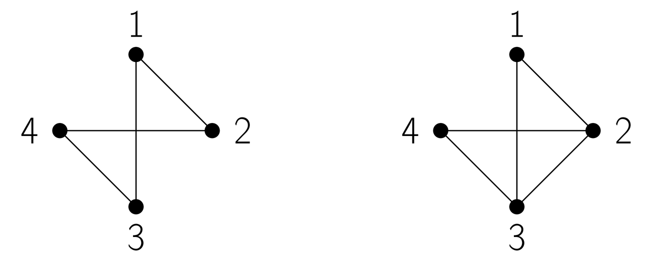
We say an NBC set is **noncrossing** if none of the edges in the NBC set cross. For G above, H_1 and H_2 are noncrossing, but H_3 is not. Let $ncnbc_k(G)$ be the number of noncrossing NBC sets of G with k edges.

Theorem Let G be an upper crossing closed (with respect to $\trianglelefteq$) graph on $[n]$ such that $\mathcal{NC}_G$ is graded. Then $\chi(\mathcal{NC}_G, t) = \sum_{k \geq 0} ncnbc_k(G) t^{n-k-1}$ where $ncnbc_k(G)$ is with respect to $\trianglelefteq$.

Perfectly Labeled Graphs

Let G be a graph on $[n]$. We say G is **perfectly labeled** if whenever $i < j < k$ and $ik, jk \in E(G)$, $ij \in E(G)$.

For example, the graph G below is not perfectly labeled, but H is perfectly labeled.



$G \quad H$

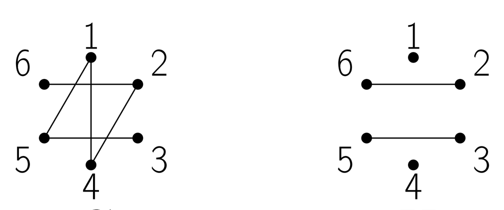
Theorem Let G be a perfectly labeled graph. Then the following hold.

1. $\mathcal{NC}_G$ is graded.
2. $\mathcal{NC}_G$ has an S_n EL-labeling and hence is shellable.
3. If G is crossing closed, then $\mathcal{NC}_G$ is a supersolvable lattice.
4. The coefficients of $\chi(\mathcal{NC}_G, t)$ count noncrossing increasing spanning forests of G.

Grading of $\mathcal{NC}_G$

The noncrossing bond poset isn't always graded. To be graded, each time we have a noncrossing bond, we must be able to merge a pair of blocks to get a noncrossing bond.

Consider the graph G below. The bond $H = 1/26/35/4$ is noncrossing. No pair of blocks can be merged together to get a noncrossing bond. Thus $\mathcal{NC}_G$ is not graded.



$G \quad H$