P-Partitions and p-positivity

How we learned to stop worrying and love α-unimodal sets

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Quasisymmetric functions

A *quasisymmetric function* f is a formal power series $f \in \mathbb{Q}[[x_1, x_2, \dots]]$ such that the degree of f is finite, and for every composition $(\alpha_1, \dots, \alpha_\ell)$ the coefficient of $\mathbf{x}_{i_1}^{\alpha_1} \cdots \mathbf{x}_{i_\ell}^{\alpha_\ell}$ in f is the same for all integer sequences $1 \leq i_1 < i_2 < \cdots < i_\ell$.

The *monomial quasisymmetric function* M_α are

$$\mathrm{M}_\alpha(\mathbf{x}) \coloneqq \sum_{i_1 < i_2 < \cdots < i_\ell} \mathbf{x}_{i_1}^{\alpha_1} \mathbf{x}_{i_2}^{\alpha_2} \cdots \mathbf{x}_{i_\ell}^{\alpha_\ell}.$$

The *fundamental quasisymmetric functions* of degree n are indexed by subsets $S \subseteq [n-1]$ and defined as

$$\mathrm{F}_{n,S}(\mathbf{x}) \coloneqq \sum_{\substack{j_1 \leq j_2 \leq \dots \leq j_n \\ i \in S \Rightarrow j_i < j_{i+1}}} \mathbf{x}_{j_1} \cdots \mathbf{x}_{j_n}.$$

Let $S_\alpha \coloneqq \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\} \subseteq [n-1]$. The expansion of fundamental quasisymmetric functions into monomial quasisymmetric functions is also given by

$$\mathrm{F}_\alpha(\mathbf{x}) = \sum_{\beta \leq \alpha} \mathrm{M}_\beta(\mathbf{x})$$

where $\leq$ denotes refinement. We then have $\mathrm{F}_\alpha = \mathrm{F}_{n,S_\alpha}$.

Quasisymmetric power-sum functions

Let $\alpha \leq \beta$ and

$$\pi(\alpha) \coloneqq \prod_{i=1}^{\ell(\alpha)} (\alpha_1 + \alpha_2 + \cdots + \alpha_i) \quad \text{and} \quad \pi(\alpha, \beta) \coloneqq \prod_{i=1}^{\ell(\beta)} \pi(\alpha^{(i)}),$$

where $\alpha^{(i)}$ is the composition of β_i induced by α. The *quasisymmetric power sum* Ψ_α is defined as

$$\Psi_\alpha(\mathbf{x}) \coloneqq z_\alpha \sum_{\beta \geq \alpha} \frac{1}{\pi(\alpha, \beta)} \mathrm{M}_\beta(\mathbf{x}),$$

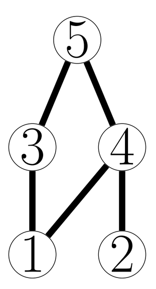
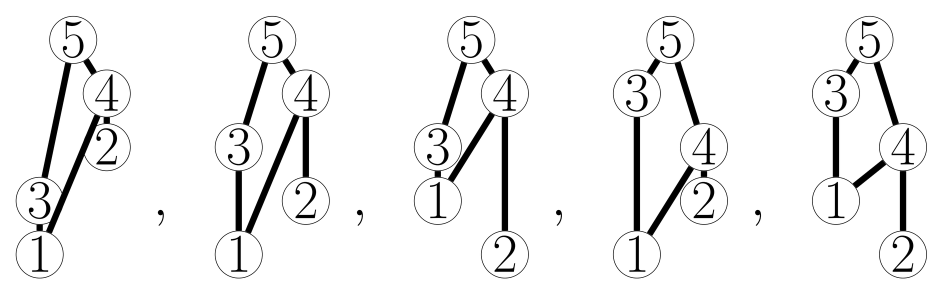
where $z_\alpha \coloneqq \prod_{i \geq 1} i^{m_i} m_i!$ and m_i denotes the number of parts of α that are equal to i. For example, $\Psi_{231} = \frac{1}{10}\mathrm{M}_6 + \frac{1}{4}\mathrm{M}_{24} + \frac{3}{5}\mathrm{M}_{51} + \mathrm{M}_{231}$. The quasisymmetric power sums refine the power sum symmetric functions as

$$\mathrm{p}_\lambda(\mathbf{x}) = \sum_{\alpha \sim \lambda} \Psi_\alpha(\mathbf{x}),$$

where the sum ranges over all compositions α whose parts rearrange to λ, see [BDH+17].

What are P-partitions?

Recall the Jordan–Hölder set of a naturally labeled poset.

$(P, w) =$  $\mathcal{L}(P, w) =$ 

13245, 12345, 21345, 12435, 21435

Let P be a poset and define

$$K_P(\mathbf{x}) \coloneqq \sum_{\substack{f : P \to \mathbb{N}^+ \\ x <_P y \Rightarrow f(x) \leq f(y)}} \prod_{x \in P} \mathbf{x}_{f(x)}.$$

For example, the P-partitions of a 3-element chain have generating function

$$K_P(\mathbf{x}) = \mathrm{h}_3(\mathbf{x}) = x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_2 x_3 + \cdots$$

Theorem (Fundamental into quasisymmetric power-sums)

Expansion of the fundamental quasisymmetric functions $\mathrm{F}_{n,S}$ into quasisymmetric power sums Ψ_α.

$$\mathrm{F}_{n,S}(\mathbf{x}) = \sum_\alpha \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} (-1)^{|S \setminus S_\alpha|}$$

Here the sum ranges over all compositions α of n such that the set S is α-unimodal.

α-unimodal sets

A permutation $\sigma \in \mathfrak{S}_n$ is *α-unimodal* if for each block $[a, b]$ of $\alpha \vDash n$ we have

$$\sigma_a > \cdots > \sigma_k < \cdots < \sigma_b.$$

for some k. For example, let $\alpha = 413$. Then the following are α-unimodal:

$$7213|4|865 \qquad 7123|5|468$$

A set $S \subseteq [n-1]$ is *α-unimodal* if it is the descent set of an α-unimodal permutation.

Theorem (P-partitions are Ψ-positive)

Let P be a finite poset on n elements. Then

$$K_P(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} |\mathcal{L}_\alpha^*(P, w)| = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} |\mathcal{O}_\alpha^*(P)|.$$

Here w denotes an arbitrary natural labeling of P, $\mathcal{L}_\alpha^*(P, w)$ consists of certain α-unimodal linear extensions of P, and $\mathcal{O}_\alpha^*(P)$ consists of certain order-preserving surjections onto chains.

α-unimodal linear extensions

Let $\mathcal{L}_\alpha(P, w)$ denote the set of all $\sigma \in \mathcal{L}(P, w)$ that are *α-unimodal*. Let

$$P_i^\alpha(\sigma) = \{x \in P : \alpha_1 + \cdots + \alpha_{i-1} < \sigma^{-1} \circ w(x) \leq \alpha_1 + \cdots + \alpha_i\}.$$

Define $\mathcal{L}_\alpha^*(P, w)$ as the set of all $\sigma \in \mathcal{L}_\alpha(P, w)$ such that each subposet $P_i^\alpha(\sigma)$ has a *unique minimal element*.

For example, if (P, w) are as before and $\alpha = (2, 3)$, then

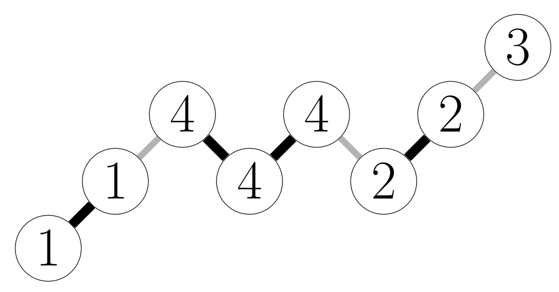
$$\mathcal{L}_{23}(P, w) = \{\hat{1}3|\hat{2}45, \hat{1}\hat{2}|\hat{3}45, \hat{2}\hat{1}|\hat{3}45, \hat{1}\hat{2}|\hat{4}35, \hat{2}\hat{1}|\hat{4}35\} \quad \text{and} \quad \mathcal{L}_{23}^*(P, w) = \{13245\},$$

where the labels of the minimal elements in each subposet are marked. Similarly, if $\alpha = (4, 1)$ then

$$\mathcal{L}_{41}(P, w) = \{\hat{1}234|\hat{5}, \hat{2}134|\hat{5}\} \quad \text{and} \quad \mathcal{L}_{41}^*(P, w) = \emptyset.$$

Order-preserving surjections

Let $\mathcal{O}_\alpha(P)$ denote the set of order-preserving surjections $f : P \to [\ell]$ such that $|f^{-1}(j)| = \alpha_j$ for $j = 1, \dots, \ell$. Example for $\alpha = (2, 2, 1, 3)$:



Let $\mathcal{O}_\alpha^*(P) \subseteq \mathcal{O}_\alpha(P)$ be the subset of surjections f such that each subposet $f^{-1}(j) \subseteq P$ has a *unique minimal element*.

Applications

The above result can be used to show (in a uniform manner) that the following quasisymmetric functions are Ψ-positive, and p-positive whenever they are symmetric.

- Chromatic (quasi)symmetric functions
- Unicellular and vertical strip LLT polynomials
- Tutte symmetric functions and B-polynomials
- Matroid quasisymmetric functions
- Certain Eulerian quasisymmetric functions

Note: One first needs to apply the involution ω that sends Ψ_α to $(-1)^{|\alpha| - \ell(\alpha)} \Psi_{\alpha^r}$ in order to get Ψ-positivity in some of the applications.

[AS] Per Alexandersson and Robin Sulzgruber. *P-partitions and p-positivity.* (2018) 1807.02460

[BDH+17] Cristina Ballantine, Zajj Daugherty, Angela Hicks, Sarah Mason, and Elizabeth Niese. *Quasisymmetric power sums.* (2017) 1710.11613

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