Characteristic elements for real hyperplane arrangements

Marcelo Aguiar *Cornell University*, Jose Bastidas† *Cornell University*, Swapneel Mahajan *Indian Institute of Technology*

Introduction

We extend the study of *characteristic elements* for real hyperplane arrangements started in [1] to the context of affine arrangements. These elements of the Tits algebra, defined by a restriction on their value on the simple characters of the algebra, determine the characteristic polynomial of the arrangement under each flat. Each arrangement possesses many characteristic elements, and the interest is in constructing particular elements from which specific information about the characteristic polynomial can be extracted. We construct a characteristic element canonically associated to each arrangement in terms of intrinsic volumes. As an application, we derive a beautiful result of Klivans and Swartz [2] relating the coefficients of the characteristic polynomial to the intrinsic volumes of the chambers.

Hyperplane arrangements

Let $\mathcal{A}$ be a finite collection of affine hyperplanes in a finite-dimensional real vector space V. We let $\Pi[\mathcal{A}]$ denote the set of *flats* and $\Sigma[\mathcal{A}]$ the set of *faces* of $\mathcal{A}$. Both form ranked posets ordered by inclusion. In addition, $\Pi[\mathcal{A}]$ is a join-semilattice with maximum element $\top = V$. The *support* of a face F is the smallest flat $\mathrm{s}(F)$ that contains it.

The *characteristic polynomial* of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) := \sum_{\mathrm{Y}} \mu(\mathrm{Y}, \top)\, t^{\mathrm{rk}(\mathrm{Y})},$$

where μ is the Mobius function of $\Pi[\mathcal{A}]$. The sum is over all flats.

The *arrangement under a flat* X is the following collection of hyperplanes in ambient space X

$$\mathcal{A}^{\mathrm{X}} = \{\mathrm{H} \cap \mathrm{X} \mid \mathrm{H} \in \mathcal{A},\ \mathrm{X} \not\subseteq \mathrm{H},\ \mathrm{H} \cap \mathrm{X} \neq \emptyset\}.$$

The Tits algebra

The set $\Sigma[\mathcal{A}]$ is a semigroup under the Tits product. Informally, the product of two faces F and G is the first face you encounter after moving a small positive distance from an interior point of F to a interior point of G, as illustrated below.

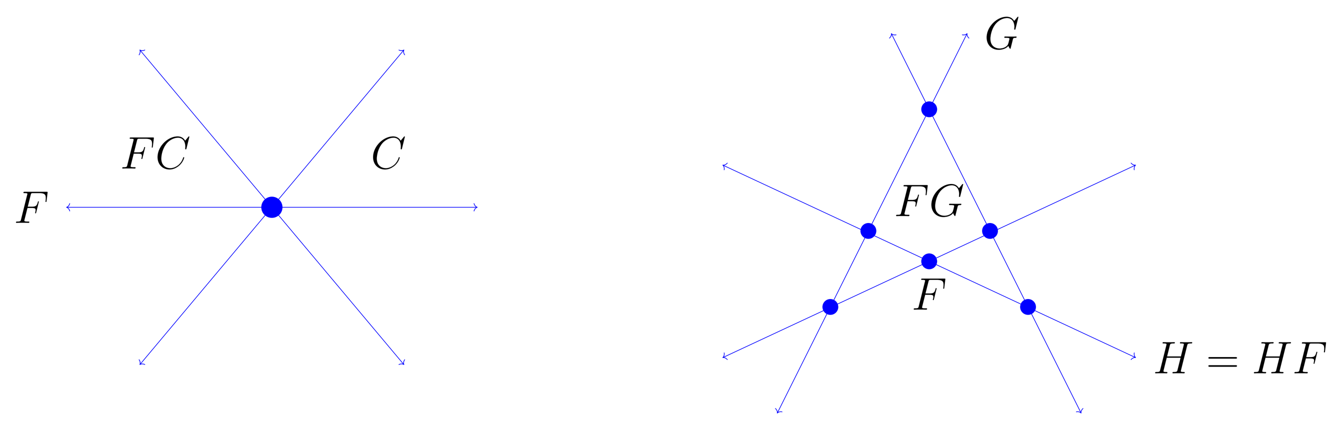

Figure 1. Product of faces in two arrangements of rank 2.

The product satisfies $\mathrm{s}(FG) = \mathrm{s}(F) \vee \mathrm{s}(G)$, where $\vee$ is the join operation of $\Pi[\mathcal{A}]$.

Let $\Bbbk$ be a field. The linearization $\Bbbk\Sigma[\mathcal{A}]$ of this semigroup is the *Tits algebra* of $\mathcal{A}$. We let $\mathtt{H}_F$ denote the basis element of $\Bbbk\Sigma[\mathcal{A}]$ associated to the face F of $\mathcal{A}$. The irreducible representations of $\Bbbk\Sigma[\mathcal{A}]$ are one-dimensional and indexed by flats. The character of the representation corresponding to a flat X is given by

$$\chi_{\mathrm{X}}(w) = \sum_{\mathrm{s}(F) \leq \mathrm{X}} w^F, \text{ where } w = \sum_F w^F \mathtt{H}_F.$$

Theorem: $\Bbbk\Sigma[\mathcal{A}]$ is a unital algebra. The unit element is

$$\upsilon = \sum_F (-1)^{\mathrm{rk}(F)} \mathtt{H}_F,$$

with F running over the set of *essentially bounded* faces of $\mathcal{A}$.

Characteristic elements

Let $t \in \Bbbk$. An element $w \in \Bbbk\Sigma[\mathcal{A}]$ is *characteristic* of parameter t if for each flat X

$$\chi_{\mathrm{X}}(w) = t^{\mathrm{rk}(\mathrm{X})}.$$

Characters are multiplicative, therefore we have.

Lemma: If u and v are characteristic elements of parameters s and t, then uv is characteristic of parameter st.

Möbius inversion implies the following result.

Lemma: An element $w \in \Bbbk\Sigma[\mathcal{A}]$ is characteristic of parameter t if and only if for every flat X,

$$\sum_{F:\, \mathrm{s}(F) = \mathrm{X}} w^F = \chi(\mathcal{A}^{\mathrm{X}}, t).$$

Let $\mathcal{A}'$ be a subarrangement of $\mathcal{A}$. There is a morphism

$$f : \Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}']$$

that sends a face F of $\mathcal{A}$ to the minimal face of $\mathcal{A}'$ that contains it. We also denote by f the linear extension $f : \Bbbk\Sigma[\mathcal{A}] \to \Bbbk\Sigma[\mathcal{A}']$.

Lemma: Let w be a characteristic element for $\mathcal{A}$ of parameter $t \neq 0$, then $t^{-\mathrm{cork}(\mathcal{A}')} f(w)$ is characteristic for $\mathcal{A}'$ of the same parameter.

Applications

Recursion: Let $\mathcal{A}'$ be a subarrangement of $\mathcal{A}$. Note that $f(F)$ is a chamber of $\mathcal{A}'$ if and only if F is not contained in any hyperplane of $\mathcal{A}'$. We conclude the following.

$$t^{\mathrm{cork}(\mathcal{A}')} \chi(\mathcal{A}', t) = \sum_{\mathrm{X}} \chi(\mathcal{A}^{\mathrm{X}}, t)$$

where the sum is taken over all flats X of $\mathcal{A}$ not contained in any hyperplane of $\mathcal{A}'$. In particular, if $\mathrm{rk}(\mathcal{A} \setminus \{\mathrm{H}\}) = \mathrm{rk}(\mathcal{A})$, then

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A} \setminus \{\mathrm{H}\}, t) - \chi(\mathcal{A}^{\mathrm{H}}, t).$$

An identity of Kung [3]: Take characteristic elements u and v of parameters s and t. Since $\mathrm{s}(FG) = \top$ if and only $\mathrm{s}(F) \vee \mathrm{s}(G) = \top$, we have

$$\chi(\mathcal{A}, st) = \sum_C (uv)^C = \sum_{\mathrm{X} \vee \mathrm{Y} = \top} \chi(\mathcal{A}^{\mathrm{X}}, s) \chi(\mathcal{A}^{\mathrm{Y}}, t)$$

Denote by $\mathcal{A}_{\mathrm{X}}$ the subarrangement of hyperplanes of $\mathcal{A}$ that contain X. An application of the previous identity gives

$$\chi(\mathcal{A}, st) = \sum_{\mathrm{X}} t^{\mathrm{rk}(\mathrm{X})} \chi(\mathcal{A}^{\mathrm{X}}, s) \chi(\mathcal{A}_{\mathrm{X}}, t).$$

Zaslavsky's formula [4]: The value of the character of a one-dimensional representation on the unit element is 1. Consequently, υ is characteristic of parameter 1. We conclude that

$$(-1)^{\mathrm{rk}(\mathcal{A})} \chi(\mathcal{A}, 1) = (-1)^{\mathrm{rk}(\mathcal{A})} \sum_C \upsilon^C$$

counts the number of essentially bounded chambers of $\mathcal{A}$.

Intrinsic elements

For each $k = 0, \ldots, n$, let $v_k(P)$ be the proportion of volume of space occupied by points that map to a k-dimensional face of P under the *nearest point* projection $\pi_P : \mathbb{R}^n \to P$.

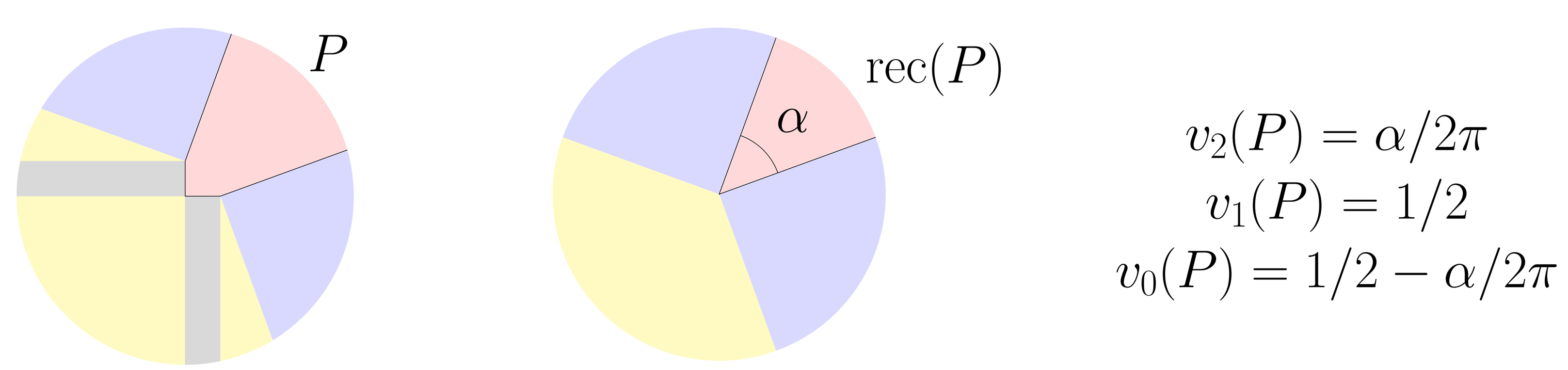

Figure 2. The k-the dimensional intrinsic volumes of P depends only on its recession cone.

$v_k(P)$ is the k-th dimensional intrinsic volume of P. Each v_k is a valuation: whenever $P \cup Q$ is convex, $v_k(P \cup Q) = v_k(P) + v_k(Q) - v_k(P \cap Q)$.

Let $\mathcal{A}$ be an arrangement and d the dimension of any minimal face of $\mathcal{A}$. The *intrinsic element* of parameter t for $\mathcal{A}$ is defined by

$$\nu_t = \sum_F (-1)^{\dim(F)} \Big(\sum_{k=d}^{\dim(F)} (-1)^k v_k(F) t^{k-d} \Big) \mathtt{H}_F.$$

Theorem: The intrinsic element ν_t is characteristic of parameter t.

***Corollary* [2]:** The coefficient of t^k in the characteristic polynomial of $\mathcal{A}$ is

$$(-1)^{\mathrm{rk}(\mathcal{A}) - k} \sum_C v_{k+d}(C).$$

References

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[3] J. P. S. Kung. A multiplication identity for characteristic polynomials of matroids. *Adv. in Appl. Math.*, 32(1-2):319–326, 2004.

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sites.google.com/view/bastidas

bastidas@math.cornell.edu