Descent Representations of Generalized Coinvariant Algebras

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Classical Coinvariant Algebras

- Let $\mathbb{Q}[x_1, \dots, x_n]$ be the polynomial ring over $\mathbb{Q}$ in n variables.
- The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{Q}[x_1, \dots, x_n]$ by permuting the indices.
- Polynomials invariant under this action are called **symmetric**.
- The invariant ideal I_n is the ideal generated by all symmetric functions with zero constant term.
- The coninvariant algebra R_n is then

$$R_n := \frac{\mathbb{Q}[x_1, \dots, x_n]}{I_n} = \frac{\mathbb{Q}[x_1, \dots, x_n]}{\langle e_n, e_{n-1}, \dots, e_1 \rangle}.$$

  Where e_i is the ith elementary symmetric function
- The quotient R_n is a graded $\mathfrak{S}_n$-module whose structure is governed by standard Young tableaux

Theorem [Chevalley] $\mathfrak{S}_n$-module isomorphism type of R_n

$$R_n \cong_{\mathfrak{S}_n} \mathbb{Q}[\mathfrak{S}_n]$$

Theorem [Lusztig-Stanley] Graded $\mathfrak{S}_n$-module isomorphism type of R_n

$$grFrob(R_n; q) = \sum_{T \in SYT(n)} q^{maj(T)} s_{shape(T)}$$

Standard Young Tableaux: Descents, Major Index, and Shape

T=

1	3	4	5
2	6	7	
8			

S=

1	2	5
3	4	8
6	7	

R=

1	3	5	7
2	4	6	8

T=

1	3	4	5
2	6	7	
8			

S=

1	2	5
3	4	8
6	7	

R=

1	3	5	7
2	4	6	8

Des(T)={1,5,7}
des(T)=3
Maj(T)=1+5+7=13
shape(T)=(4,3,1)

Des(S)={2, 5}
des(S)=2
Maj(S)=2+5=7
shape(S)=(3,3,2)

Des(R)={1,3,5,7}
des(R)=4
Maj(R)=1+3+5+7=16
shape(R)=(4,4)

Generalized Coinvariant Algebras

- Motivated by the Delta Conjecture in the field of Macdonald polynomials a generalized coinvariant algebra $R_{n,k}$ was defined by Haglund, Rhoades, and Shimozono as follows:

$$R_{n,k} = \frac{\mathbb{C}[x_1, \dots, x_n]}{\langle e_n, e_{n-1}, \dots, e_{n-k+1}, x_1^k, x_2^k, \dots, x_n^k \rangle}.$$

- The quotient $R_{n,k}$ is a graded $\mathfrak{S}_n$-module whose structure is governed by ordered set partitions of n with k blocks.

Generalized Coinvariant Algebra Results

Theorem [Haglund, Rhoades, Shimozono] Graded and ungraded $\mathfrak{S}_n$-module isomorphism types of $R_{n,k}$

$$R_{n,k} \cong_{\mathfrak{S}_n} \mathbb{Q}[\mathcal{OP}_{n,k}],$$

where $\mathcal{OP}_{n,k}$ is the set of ordered set partitions of n with k blocks.

$$grFrob(R_{n,k}; q) = \sum_{T \in SYT(n)} q^{maj(T)} \begin{bmatrix} n - des(T) - 1 \\ n-k \end{bmatrix}_q s_{shape(T)}$$

Refining by Partitions

- Dominance order: for two partitions of n, $\lambda \trianglelefteq \mu$ if $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ for all k.
- Letting $\lambda(m)$ be the exponent partition of a monomial m, define

$$P_{\trianglelefteq\mu} := \text{span}\{m \in \mathbb{C}[x_1, \dots, x_n] : \lambda(m) \trianglelefteq \mu\}$$

$$P_{\triangleleft\mu} := \text{span}\{m \in \mathbb{C}[x_1, \dots, x_n] : \lambda(m) \triangleleft \mu\}$$

- Let the projections of $P_{\trianglelefteq\mu}$ and $P_{\triangleleft\mu}$ onto R_n be $Q_{\trianglelefteq\mu}$ and $Q_{\triangleleft\mu}$, and then define

$$R_{n,\lambda} := Q_{\trianglelefteq\lambda}/Q_{\triangleleft\lambda}$$

- Then

$$R_n = \bigoplus_{\lambda} R_{n,\lambda}$$

- Furthermore the degree d component of R_n is isomormorphic to

$$\bigoplus_{\lambda \vdash d} R_{n,\lambda}$$

- We can define $R_{n,k,\lambda}$ analogously, and it will refine the grading of $R_{n,k}$.

Refinement Results

Theorem [Adin, Brenti, Roichman]
$R_{n,\lambda} = 0$ unless $\lambda_i - \lambda_{i+1} = 0, 1$ for all i, and $\lambda_n = 0$, and in this case

$$Frob(R_{n,\lambda}) = \sum_{\mu \vdash n} c_{\lambda,\mu} s_\mu$$

where $c_{\lambda,\mu} =: \{T \in SYT(\mu) : Des(T) = Des(\lambda)\}|$

Theorem [M] $R_{n,k,\lambda} = 0$ unless $\lambda_1 < k$, $\lambda_n = 0$, and $\lambda_i - \lambda_{i+1} = 0, 1$ for $i > n-k$, and in this case

$$Frob(R_{n,k,\lambda}) = \sum_{\mu \vdash n} c_{\lambda,\mu} s_\mu$$

$$c_{\lambda,\mu} = |\{T \in SYT(\mu) : Des_{n-k+1,n}(\lambda) \subseteq Des(T) \subseteq Des(\lambda)\}|$$

Example Calculations

Example for $R_{n,\lambda}$:
Let $n = 8$, $\lambda = (4, 4, 3, 2, 2, 1, 1)$, and $\mu = (3, 3, 2)$

$Des(\lambda) = \{2, 3, 5, 7\}$

1	2	7
3	5	8
4	6	

1	2	5
3	6	7
4	8	

$c_{\lambda,\mu} = 2$

Example for $R_{n,k,\lambda}$:
Let $n = 8$, $k = 6$, $\lambda = (5, 5, 2, 2, 1, 1, 1)$, and $\mu = (4, 3, 1)$

$Des_{3,8}(\lambda) = \{4, 7\}$, $Des(\lambda) = \{2, 4, 7\}$

1	2	4	7
3	5	6	
8			

1	2	4	7
3	6	8	
5			

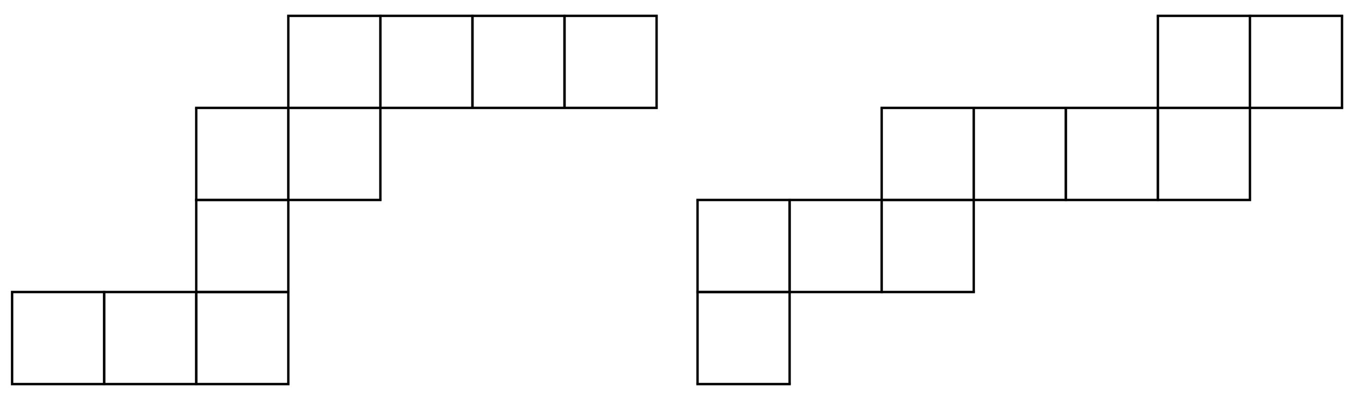
1	2	6	7
3	4	8	
5			

1	2	3	4
5	6	7	
8			

$c_{\lambda,\mu} = 4$

Ribbon Tableaux and Crystals

- Using the Robinson-Schensted-Knuth algorithm, this result can be expressed in terms of ribbon tableaux, which are connected skew tableaux that contain no 2×2 boxes.



- Specifically, for each λ, there is a partition ρ, and a ribbon shape γ, such that

$$\text{Frob}(R_{n,k,\lambda}) = s_\gamma h_\rho. \qquad (1)$$

  The ρ is determined by the differences between optional descents, and the γ is determined by the differences between mandetory descents. For example if we had optional descents $\{1, 3\}$, and mandetory descents $\{6, 7, 9\}$, we would have $\rho = (2, 1)$ and γ would be the left ribbon shape above.
- Benkart, Colmenarejo, Harris, Orellana, Panova, Schilling, and Yip defined a crystal structure on ordered multiset partitions using the minimaj statistic such that when graded by minimaj, the character of their crystal is

$$(rev_q \circ \omega) grFrob(R_{n,k}).$$

- This crystal is built up from smaller crystals that have characters equal to

$$s_\gamma e_\rho$$

  for a ribbon shape γ and a partition ρ.
- Therefore up to the right choice of λ, $R_{n,k,\lambda}$ corresponds to these smaller crystal.

References

References

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