Modifying Curtiss' theorem to prove CLTs

Gene B. Kim† *University of Southern California* and Sangchul Lee *University of California – Los Angeles*

Permutation Statistics

Let $\pi \in S_n$ be a permutation.

- π has a *descent* at position i if $\pi(i) > \pi(i+1)$. The *descent number* of π, $d(\pi)$, is the number of descents.
- The *major index* of π, $maj(\pi)$, is the sum of positions at which π has a descent.
- π has a *peak* at position i if $\pi(i-1) < \pi(i)$ and $\pi(i) > \pi(i+1)$. We use $p(\pi)$ to denote the number of peaks.

It is widely known that $d(\pi)$ is asymptotically normal if π is chosen uniformly at random from S_n. Fulman showed that $d(\pi)$ is asymptotically normal if π is chosen uniformly at random from large conjugacy classes of S_n.

CLTs for Descents

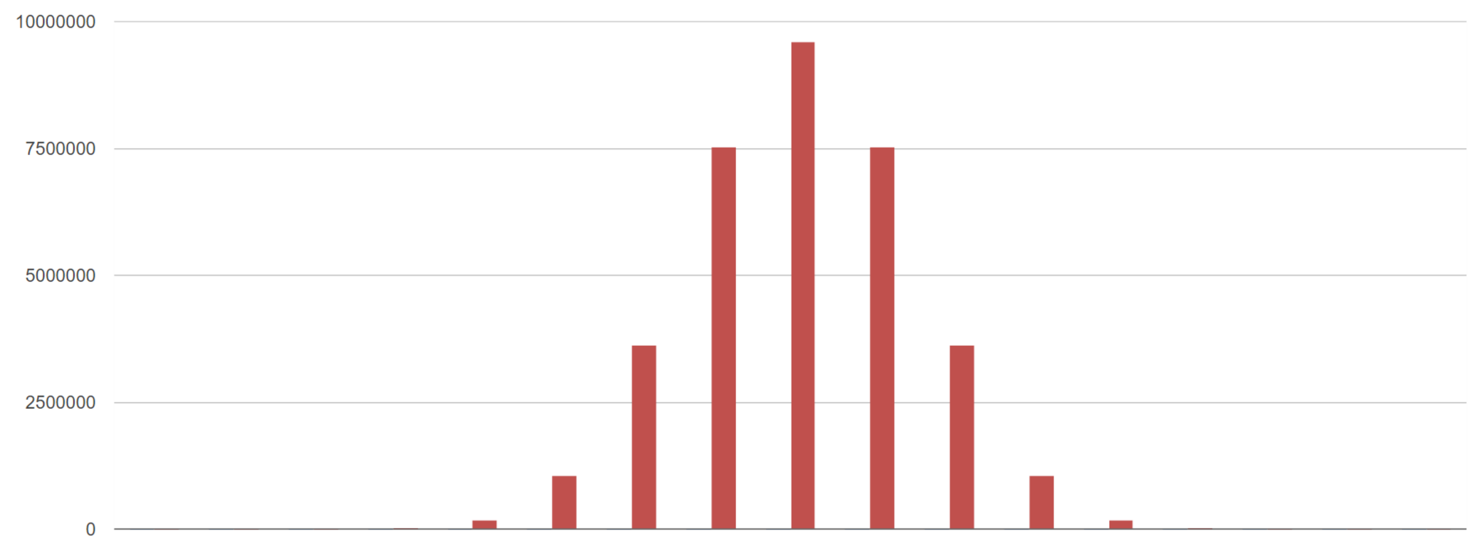

Figure 1. Descents in fixed point free involutions of S_{16}

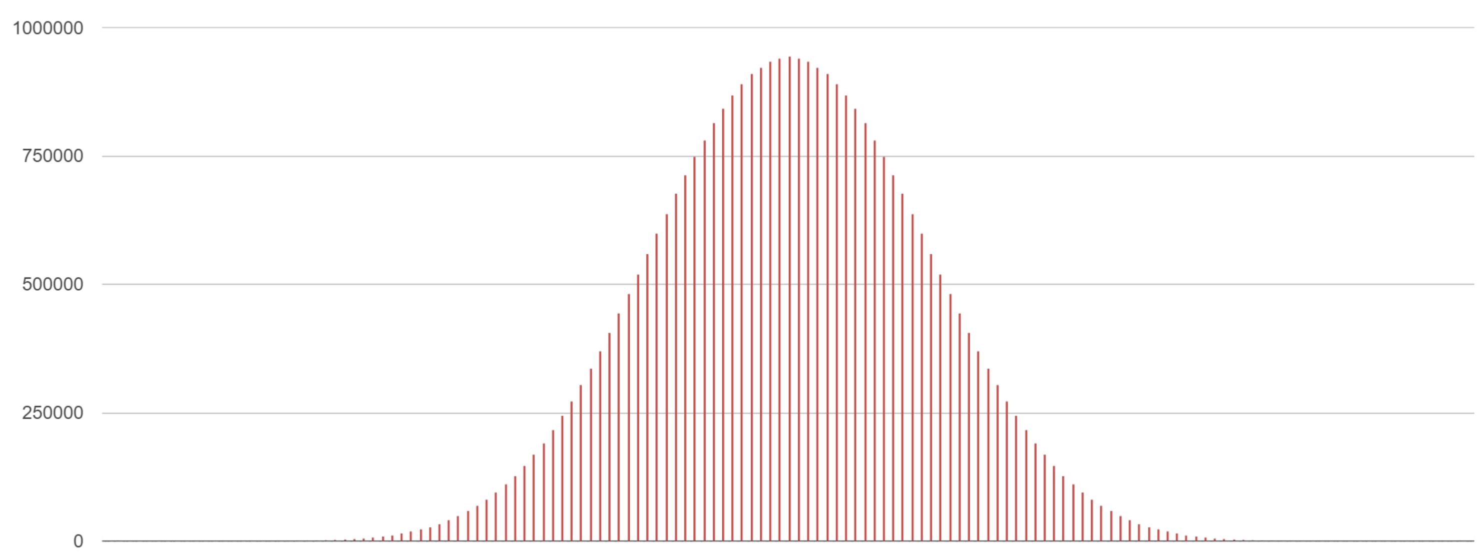

Figure 2. Major indices in fixed point free involutions of S_{16}

Theorem (Curtiss 1942)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables and let $M_n(s)$ be the MGF (moment generating function) of X_n. If $\{M_n(s)\}_{n=1}^{\infty}$ converges pointwise to some function $M(s)$, then $M(s)$ is the MGF of some random variable X, and

$$X_n \Rightarrow X.$$

In 1997, Fulman calculated the generating function for descents in particular conjugacy classes of S_n, which is convergent for $|t| < 1$. From this, one can calculate the MGF for descents which is convergent for $s < 0$. Kim and Rhoades proved, in 2015, that the descents and major indices of fixed point free involutions are symmetric. Hence, the MGF of descents in fixed point free involutions is symmetric for all s, and so, by Curtiss' theorem, we showed:

Theorem (K 2017)

For every $n \geq 1$ even, let C_n be the conjugacy class of fixed point free involutions in S_n. Then, the distribution of $d(\pi)$ in C_n is asymptotically normal with mean $\frac{n}{2}$ and variance $\frac{n}{12}$.

To extend the result to all conjugacy classes of S_n, Kim and Lee derived a version of Fulman's generating function for descents that is convergent for $|t| > 1$ and used Curtiss' theorem to show:

Theorem (K-L 2018)

Let C_n be a conjugacy class of S_n for each $n \geq 1$ and let $\alpha(C_n)$ be the density of fixed points of C_n. Suppose that $\alpha(C_n) \to \alpha \in [0,1]$. Then, the distribution of $d(\pi)$ for π drawn uniformly at random from C_n is asymptotically normal with asymptotic mean $\left(\frac{1-\alpha^2}{2}\right)n$ and asymptotic variance $\left(\frac{1-4\alpha^3+3\alpha^4}{12}\right)n$.

CLT for Descents and Major Indices

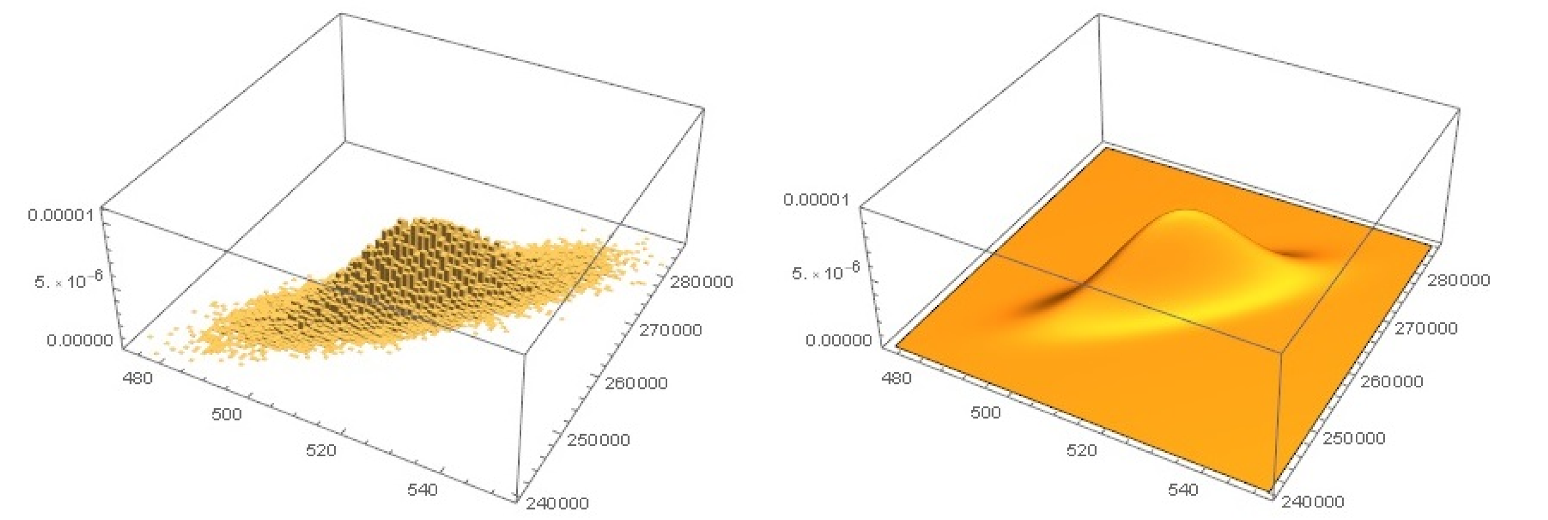

Figure 3. Descents and major indices of 100,000 permutations drawn from $C_{2^{500}} \subset S_{1000}$

There is a MGF $M(s,r)$ for the joint distribution of descents and major indices for a particular conjugacy class of S_n, which is convergent for $s, r < 0$. If we were to take a similar approach with Curtiss' theorem to prove a central limit theorem, we need three alternate expansions that converge on the other quadrants. We sidestepped this tedious problem by proving a modification of Curtiss' theorem:

Theorem (K-L 2018)

Let X_n be random vectors in $\mathbb{R}^d$ for each $n \in \mathbb{N} \cup \{\infty\}$. Suppose that there is a non-empty subset $U \subseteq \mathbb{R}^d$ such that $M_{X_n}(s) \to M_X(s)$ pointwise for all $s \in U$. Then, X_n converges in distribution to X_∞.

By excluding the origin from the region of convergence, we lose the guarantee that the limit is a MGF. However, if we show that the MGF's converge pointwise to a desired MGF, we do not need to worry about this!

Theorem (K-L 2018)

Let C_n be a conjugacy class of S_n for each $n \geq 1$ and let $\alpha(C_n)$ be the density of fixed points of C_n. Suppose that $\alpha(C_n) \to \alpha \in [0,1]$. Then, the distribution of $(d(\pi), maj(\pi))$ for π drawn uniformly at random from C_n is asymptotically bivariate normal with asymptotic mean $\left(\left(\frac{1-\alpha^2}{2}\right)n, \left(\frac{1-\alpha^2}{4}\right)n^2\right)$ and asymptotic covariance matrix

$$\begin{pmatrix} \left(\frac{1-4\alpha^3+3\alpha^4}{12}\right)n & \left(\frac{1-4\alpha^3+3\alpha^4}{24}\right)n^2 \\ \left(\frac{1-4\alpha^3+3\alpha^4}{24}\right)n^2 & \frac{1-\alpha^3}{36}n^3 \end{pmatrix}$$

CLT for Peaks

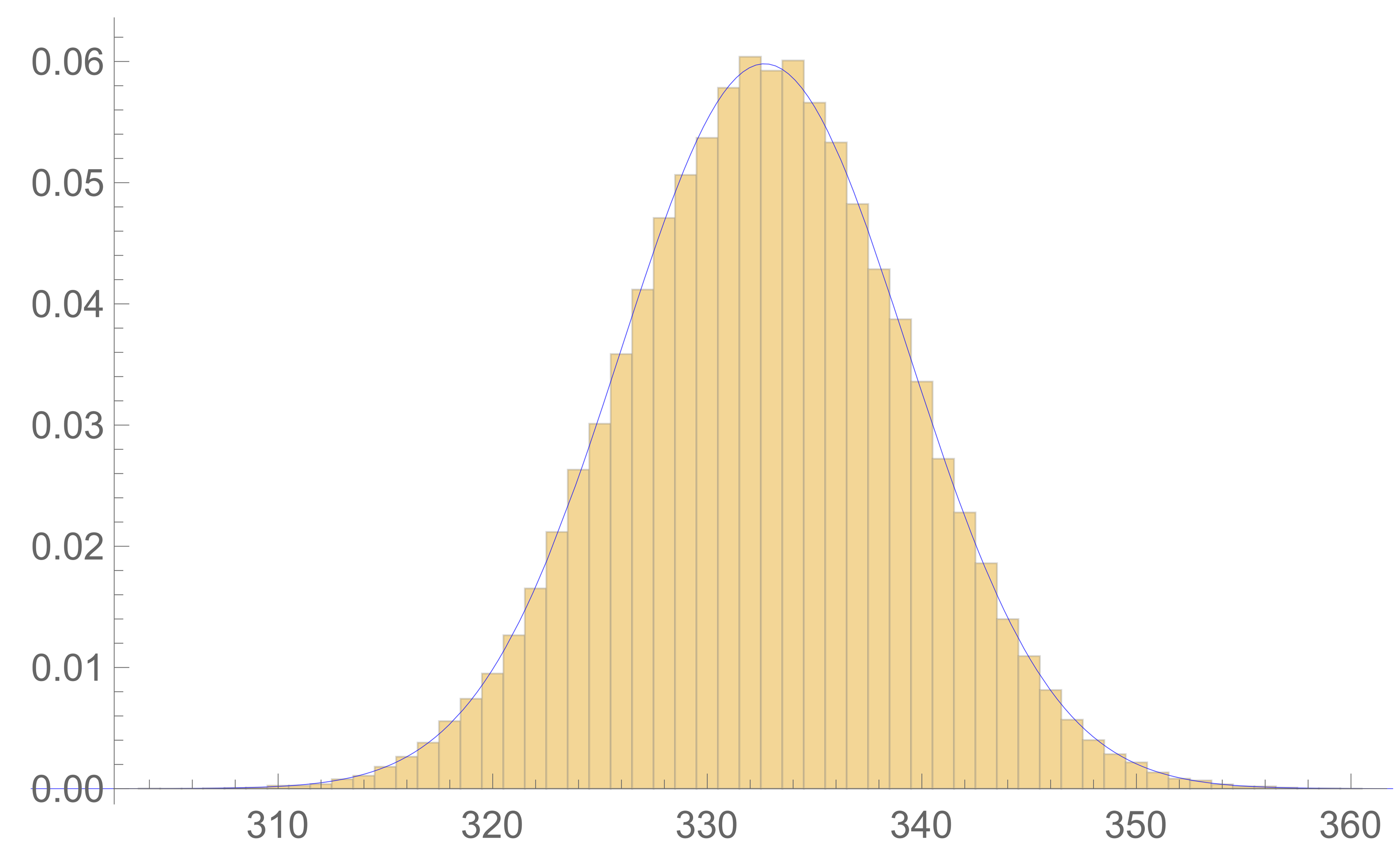

Figure 4. Peaks of 100,000 permutations drawn from $C_{2^{250}4^{125}} \subset S_{1000}$

Fulman and the authors recently used the modified Curtiss' theorem to prove a CLT for peaks:

Theorem (Fulman-K-L 2019)

Let C_n be a conjugacy class of S_n for each $n \geq 1$ and let $\alpha(C_n)$ be the density of fixed points of C_n. Suppose that $\alpha(C_n) \to \alpha \in [0,1]$. Then, the distribution of $p(\pi)$ for π drawn uniformly at random from C_n is asymptotically normal with asymptotic mean $\left(\frac{1-\alpha^3}{3}\right)n$ and asymptotic variance $\left(\frac{2}{45} + \frac{1}{9}\alpha^3 - \frac{3}{5}\alpha^5 + \frac{4}{9}\alpha^6\right)n$.

http://www.genebkim.com genebkim@usc.edu